The Borel Complexity of Isomorphism

for some Ordered Theories

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We then ask if this theorem can be extended to more general contexts.

Roadmap

1. O-Minimal Theories
2. Colored Linear Orders
3. Extensions
   - Closed Questions
   - Open Questions

The Nonstructure Hypothesis

Let T be an o-minimal theory.

A nonsimple type $p \in S_1(A)$ is a nonalgebraic type where there is a set $B \subset p(\mathfrak{C})$ and an element $b \in p(\mathfrak{C})$ where $b \in \mathrm{cl}(AB)$ but $b \notin B$.

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Our nonstructure hypothesis is the existence of a nonsimple type.

Proposition

There is a nonsimple type over some set iff there is one over the empty set.

Theories with Structure, I

L. Mayer (1988) showed Vaught's conjecture holds for o-minimal theories with the following:

Lemma

Say T has no nonsimple types. Then $M \cong N$ if and only if, for every $p \in S_1(\emptyset)$, $(p(M), <) \cong (p(N), <)$.

Lemma

Say $p \in S_1(\emptyset)$ is not nonsimple.
Then there are at most six choices for the order type of $p(M)$.
If p is isolated, there is only one.

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4. If $\kappa = 2^{\aleph_0}$, then $\cong_T$ is $\cong_2$

Archimedean Equivalence

Let $p \in S_1(\emptyset)$ be a type. Define $\sim$ on $p(\mathfrak{C})$ where, for $a, b \in p(\mathfrak{C})$,

$a \sim b$ iff there are $a_1, a_2 \in \mathrm{cl}^p(a)$ with $a_1 \leq b \leq a_2$

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$\sim$ *is an equivalence relation on* $p(\mathfrak{C})$ *with convex classes*

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The other axioms are similarly verified.

Theories without Structure: Faithful Types

$p \in S_1(\emptyset)$ is faithful if, for all sets of pairwise $\sim$-inequivalent $A \subset p(\mathfrak{C})$,
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- Let M_I be prime over A_I
- The map $I \mapsto M_I$ is Borel
- $p(M_I)/\sim$ has order type $(I, <)$, so this is a Borel reduction

Nonisolated Types, I

Proposition

Non-cuts are faithful.

A proof:

- Pick a minimal counterexample $c < b_1 < \cdots < b_{n+1}$ where $f(\overline{b}, b_{n+1}) = c$ but $c \ll b_1 \ll \cdots \ll b_{n+1}$

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- $\mathrm{tp}(c/\overline{b})$ is a cut or non-cut (respectively)
- No such definable function exists (continuity-monotonicity theorem)

Nonisolated Types, II

By similar logic, nonisolated nonsimple types always lead to faithful types:

- Nonsimple non-cuts are always faithful
- Nonsimple cuts can be faithful
- If a nonsimple cut is unfaithful, there is a nonsimple non-cut "nearby"

So that:

Proposition

If T has a nonisolated nonsimple type over $\emptyset$, then $\cong_T$ is Borel complete.

Isolated Types, Example

Example

Let $M = (\mathbb{Q}, <, f)$, where $f(x, y, z) = x + y - z$.

Then $T = \mathrm{Th}(M)$ has only one 1-type $(x = x)$ and no unary functions, but has a binary function $(x, y) \mapsto 2x - y$. It is unfaithful.

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An idea! If we add constants for "zero" and "one," the resulting type $\{x > n : n \in \omega\}$ is a (faithful) non-cut with a unary function $x \mapsto 2x$.

Adding Parameters

Let $p \in S_1(\emptyset)$ be n-nonsimple, isolated. Let $\overline{a} = a_1 < \cdots < a_n$ be from p.

Then $q \in S_1(\overline{a})$, given by "x realizes p and $x > \mathrm{cl}^p(\overline{a})$" is a nonsimple (faithful) non-cut.

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Problem: If we compute M_I as before, the ladder $q(M_I)/\sim$ is isomorphic to I, but is not preserved under isomorphism.

Note that $\cong$ for $T_{\overline{a}}$ is Borel complete.

A Canonical Tail

Lemma

Suppose $\overline{a}$ and $\overline{b}$ are n-tuples from p, and c, d are realizations of p. If $c, d > cl(\overline{a}\overline{b})$, then $c \sim_{\overline{a}} d$ if and only if $c \sim_{\overline{b}} d$.

Thus, $(p_{\overline{a}}(M), <)$ and $(p_{\overline{b}}(M), <)$ are isomorphic on a tail.

Therefore: if $M_I \cong M_J$, then $(I, <)$ and $(J, <)$ have an isomorphic tail.

A Nice Set of Linear Orders

Lemma

There is a Borel function $f : LO \to LO$ where for all $I, J \in LO$, TFAE:

- $I \cong J$
- $f(I) \cong f(J)$
- $f(I)$ *and* $f(J)$ *are isomorphic on a tail*

For the curious:

- Let $(X, <)$ be $\{0\} \cup \{q \in \mathbb{Q} : 1 \leq q \leq 2\} \cup \{3\}$.
- The map is $I \mapsto \omega \times [(I \times X) \cup \{\infty\}]$

Completing the Proof

This gives us our final theorem:

Theorem

Suppose T is o-minimal with a nonsimple type.
Then $\cong_T$ is Borel complete.

A proof:

- Let $I \in \mathrm{LO}$; let $f(I)$ be as in the lemma

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- Then $p(M_I)/\sim_{\overline{a}}$ is $(f(I), <)$
- If $M_I \cong M_J$, then $(f(I), <)$ and $(f(J), <)$ are isomorphic on a tail
- So $I \mapsto M_{f(I)}$ is a Borel reduction

Recap

What we showed:

If T has a nonsimple type, then

- $\cong_T$ is Borel complete

If T has no nonsimple type, then

- If $\kappa = 0$, then $\cong_T$ is $(1, =)$
- If $1 \leq \kappa < \aleph_0$, then $\cong_T$ is $(n, =)$ for some $3 \leq n < \omega$
- If $\kappa = \aleph_0$, then $\cong_T$ is $\cong_1$ (reals)
- If $\kappa = 2^{\aleph_0}$, then $\cong_T$ is $\cong_2$ (countable sets of reals)

where κ is the number of independent nonisolated types in $S_1(\emptyset)$.

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Colored Linear Orders

A typical language is a language $L = \{<\} \cup \{P_n : n < \kappa\}$ for some $\kappa \leq \aleph_0$.

A typical theory is any complete L-theory T where $<$ is a linear order.

Theorem (M. Rubin)

Typical theories satisfy Vaught's conjecture. In particular:
If T is typical, then T has finitely many or continuum-many models.
If L is finite, T is $\aleph_0$-categorical or has continuum-many models.

Extensions

Rubin's proof has been ripe for generalizations:

Corollary (Wagner, 1979)

Typical theories satisfy Martin's conjecture.

Corollary (Schirmann, 1997)

Complete theories of linear orders are $\aleph_0$-categorical or Borel complete.

Corollary (R.)

If T is typical, then $\cong_T$ is one of:

$(1, =)$, $(n, =)$, $\cong_1$, $\cong_2$, *or Borel complete.*

If L is finite, T is $\aleph_0$-categorical or Borel complete.

Convex Types

A convex formula $\phi(x, \overline{a})$ is one whose set of realizations is convex.

A convex type is a complete consistent set of convex formulas.

Let $IT(T)$ be the space of convex types over $\emptyset$.

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Definition / Theorem (Rubin)

The following are equivalent for $\mathcal{I} = (I, <, P_n)_{n \in \kappa}$ with two or more points:

- *$\mathcal{I}$ has no proper definable convex subsets*
- *The canonical embeddings $\mathcal{I} \to \mathcal{I} + \mathcal{I}$ are elementary*
- *The canonical embeddings $\mathcal{I} \to \sum_{x \in X} \mathcal{I}$ are elementary*
- *The above, but for any $\mathcal{J} \equiv \mathcal{I}$*

Call such an $\mathcal{I}$ self-additive.

A Condensation?

Let $\mathcal{I}$ be typical. Say $a \sim b$ if there is a $\phi(x, y)$ such that:

- $\phi(I, a)$ is convex and bounded
- $\mathcal{I} \models \phi(a, a) \wedge \phi(b, a)$

Example: In $(L \times \mathbb{Z}, <)$, $a/\sim$ is $\{S^n(a) : n \in \mathbb{Z}\}$
Example: $\sim$ is not symmetric on (e.g.) $(\omega + \mathbb{Z}, <)$

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- $\tau(z, a)$ includes a and c and is convex
- $\tau(z, a)$ is bounded, as witnessed by $I \prec I + I + I$ (SA)

Dichotomy for Self-Additive Orders, I

Lemma

Suppose $\mathcal{I} = (I, <, P_n)_{n \in \kappa}$ is self-additive and $S_1(\emptyset)$ is infinite. Then $Th(\mathcal{I})$ is Borel complete.

Sketch of the proof:

- Let $p \in S_1(\emptyset)$ be nonisolated

Lemma

Suppose $\mathcal{I} = (I, <, P_n)_{n \in \kappa}$ is self-additive and $S_1(\emptyset)$ is infinite.
Then $Th(\mathcal{I})$ is Borel complete.

Sketch of the proof:

- Let $p \in S_1(\emptyset)$ be nonisolated
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- Then $\mathcal{C}$ has exactly one $\sim$-class containing a realization of p...
- ...and $L \mapsto L \times \mathcal{C}$ is a Borel reduction $\mathrm{LO} \to \mathrm{Mod}(T)$

Dichotomy for Self-Additive Orders, II

Lemma

Suppose $\mathcal{I}$ is typical and $S_1(\emptyset)$ is finite. Then $Th(\mathcal{I})$ is $\aleph_0$-categorical or Borel complete.

So if $\mathcal{I}$ is self-additive, then $\mathrm{Th}(\mathcal{I})$ is Borel complete or $\aleph_0$-categorical.

The General Case, I

If $\mathfrak{C} \equiv \mathcal{I}$ is $\aleph_0$-saturated, then for every $\Phi \in IT(T)$, $\Phi(\mathfrak{C})$ is self-additive.

Proposition

Let $\mathcal{M} \equiv \mathcal{N}$ be typical. Then $\mathcal{M} \cong \mathcal{N}$ if and only if, for every $\Phi \in IT(T)$, $\Phi(\mathcal{M}) \cong \Phi(\mathcal{N})$.

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Proposition

If $Th(\Phi(\mathfrak{C}))$ is Borel complete for some Φ, then $Th(\mathcal{I})$ is Borel complete.

Proof: Essentially, put models of $\mathrm{Th}(\Phi(\mathfrak{C}))$ into an (otherwise unchanged) model of $\mathrm{Th}(\mathcal{I})$.

The General Case, II

Proposition

For all $\mathcal{M} \prec \mathfrak{C}$, all $\Phi \in IT(T)$, there is $\mathcal{N}$ where $\Phi(\mathcal{M}) \prec \mathcal{N}$ and $\mathcal{N}$ is a convex subset of $\Phi(\mathfrak{C})$.

Lemma (Rosenstein; Mwesigye / Truss)

Let $\mathcal{I}$ be countable and $\aleph_0$-categorical. There are only finitely many convex subsets of $\mathcal{I}$ up to isomorphism.

Proposition

If $\Phi \in IT(T)$ is isolated and $\Phi(\mathfrak{C})$ is not Borel complete, there is only one choice for $\Phi(\mathcal{M})$ up to $\cong$.

The General Case, III

Let T be a typical theory. Say T is locally easy if, for all $\Phi \in IT(T)$, $\mathrm{Th}(\Phi(\mathfrak{C}))$ is $\aleph_0$-categorical.

Theorem

If T is not locally easy, T is Borel complete.
If T is locally easy, then $\cong_T$ is:

- $(1, =)$, *if* $\kappa = 0$
- $(n, =)$, *for some* $3 \leq n < \omega$, *if* $1 \leq \kappa < \aleph_0$
- $\cong_1$, *if* $\kappa = \aleph_0$
- $\cong_2$, *if* $\kappa = 2^{\aleph_0}$

where κ is the number of nonisolated convex types.

Note that convex types are always independent.

Roadmap

1. O-Minimal Theories
2. Colored Linear Orders
3. Extensions
   - Closed Questions
   - Open Questions

Possible Similarities, I

How strong is the analogy between the two cases?

Theorem

If T is a colored linear order in a finite language, T is $\aleph_0$-categorical or Borel complete.

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The analogous statement is not true for o-minimal theories:

Example

Let $\mathcal{M} = \left(\mathbb{R}^{\mathrm{alg}}, <, f, g\right)$, where $f(x) = x + 1$, $g(x) = x + \sqrt{2}$, and both are restricted to $[0, 2]$.

$T = \mathrm{Th}(\mathcal{M})$ is not small – $\mathrm{cl}(\emptyset)$ has a perfect subset – but T has no nonsimple types, so is not Borel complete. So $\cong_T$ is $\cong_2$.

Possible Similarities, II

How strong is the analogy between the two cases?

Theorem

Let T be a Borel complete o-minimal theory. Then some restriction of T to a finite language is Borel complete.

Possible Similarities, II

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Theorem

Let T be a Borel complete o-minimal theory. Then some restriction of T to a finite language is Borel complete.

The analogous statement is not true for colored linear orders:

Example

Let T say $<$ is dense without endpoints, and the P_n are disjoint and dense in the order for all $n \in \omega$.

Then T is Borel complete – the set of "uncolored" elements can have any order type – but every restriction of T to a finite language is $\aleph_0$-categorical.

Infinitary Logic?

All the theorems stated only work for complete first-order theories.
Do they apply for $L_{\omega_1,\omega}$-sentences? If not, why not?

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Do they apply for $L_{\omega_1,\omega}$-sentences? If not, why not?

Theorem (Steel)

Let $L = \{<\}$ and let $\Phi \in L_{\omega_1,\omega}$ be a sentence whose models are all trees. Then Φ satisfies Vaught's conjecture.

The proof does not give rise to a structure theory for models of Φ.

Working with Trees

What if we generalize from linear orders to trees? Do we get the same theorem? Is there a similar proof?

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Unknown, but two relevant theorems:

- Steel (1978): Complete theories of trees satisfy Vaught's conjecture.
- Barham (2015) gave a characterization of $\aleph_0$-categorical $\aleph_0$-colored trees in the same flavor as Rosenstein's.

Ordered Theories, I

Let $L = \{<, \ldots\}$ and T be a complete theory making $<$ a linear order.

Question

Must $\cong_T$ be among $(1, =)$, $(n, =)$, $\cong_1$, $\cong_2$, or be Borel complete?

The answer is almost certainly no, but what would an example look like?

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The answer is almost certainly no, but what would an example look like?

Proposition

Let T be an ordered theory. Let $L' = \{E\} \cup L$, and let T' be any complete theory stating:

- *$<$ is a linear order*
- *E is an equivalence relation with infinitely many classes, all convex*
- *The E-classes are independent models of T*

Then T' is either $\aleph_0$-categorical or Borel complete.

So the usual method of getting "jumps" doesn't work here.

Ordered Theories, II

Let $L = \{<, \ldots\}$ and T be a complete theory making $<$ a linear order.

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Question

Must $\cong_T$ be among $(1, =)$, $(n, =)$, $\cong_1$, $\cong_2$, or be Borel complete?

Supposing we wanted to imitate the previous proofs. The most important ingredient on the non-structure side is a definable, convex equivalence relation within convex types.

Question

Are there natural conditions on T which produce a definable convex equivalence relation within types (besides $x = x$)?

If so, we can "probably" do some omitting types magic and produce interesting quotient orders.