Potential Cardinality
for Countable First-Order Theories

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The Main Idea

The Goal: Understand the countable models of a theory Φ

Chosen framework: if $\Phi \leq_B \Psi$ then the countable models of Φ are "more tame" than the countable models of Ψ.

Relatively easy: show $\Phi \leq_B \Psi$;
Relatively hard: show $\Phi \not\leq_B \Psi$

Theorem (Ulrich, R., Laskowski)

If $\Phi \leq_B \Psi$ then $\|\Phi\| \leq \|\Psi\|$.

Motivation?

Why study Borel reductions?

Comparing the number of models is pretty coarse. Consider:

1. Countable sequences of $\mathbb{Q}$-vector spaces
2. Graphs

These both have $\beth_1$ countable models, but
Borel reductions can easily show the former is much smaller than the latter.

Counterexamples to Vaught's conjecture are pretty weird;
Borel reductions give a nice way to make this formal (even given CH).

Borel Reductions

Fix $\Phi, \Psi \in L_{\omega_1\omega}$.

$\mathrm{Mod}_\omega(\Phi)$ and $\mathrm{Mod}_\omega(\Psi)$ are Polish spaces under the formula topology.

$f : \mathrm{Mod}_\omega(\Phi) \to \mathrm{Mod}_\omega(\Psi)$ is a Borel reduction if:

1. For all $M, N \models \Phi$, $M \cong N$ iff $f(M) \cong f(N)$
2. For any $\psi \in L_{\omega_1\omega}$ (with parameters from ω)
   there is a $\phi \in L_{\omega_1\omega}$ (with parameters from ω)
   where $f^{-1}(\mathrm{Mod}_\omega(\Psi \wedge \psi)) = \mathrm{Mod}_\omega(\Phi \wedge \phi)$

   (preimages of Borel sets are Borel)

Say $\Phi \leq_B \Psi$.

A Serious Question

It's somewhat clear how to show that $\Phi \leq_B \Psi$.

How is it possible to show that $\Phi \not\leq_B \Psi$?

Partial answer: there are some techniques, but they only apply when Φ and/or Ψ is Borel[1] (and low in the hierarchy).

Very little is known when you can't assume Borel.

[1] That is, the isomorphism relation is a Borel subset of the product space.

Roadmap

1. Borel Reductions
2. Model Theory and Games
3. Connections

Back-and-Forth Equivalence

Let M and N be L-structures. $\mathcal{F} : M \to N$ is a back-and-forth system if:

1. $\mathcal{F}$ is a nonempty set of partial functions $M \to N$
2. All $f \in \mathcal{F}$ preserve L-atoms and their negations
3. For all $f \in \mathcal{F}$, all $m \in M$, and all $n \in N$,
   there is a $g \in \mathcal{F}$ where $m \in \mathrm{dom}(g)$, $n \in \mathrm{im}(g)$, and $f \subset g$

Say $M \equiv_{\infty\omega} N$ if there is such an $\mathcal{F}$.

If $M \cong N$ then $M \equiv_{\infty\omega} N$.
If M and N are countable and $M \equiv_{\infty\omega} N$, then $M \cong N$.

Canonical Scott Sentences

Canonical Scott sentences form a canonical invariant of each $\equiv_{\infty\omega}$-class.

For all M, N, the following are equivalent:

1. $M \equiv_{\infty\omega} N$
2. $\mathrm{css}(M) = \mathrm{css}(N)$
3. $N \models \mathrm{css}(M)$ (and/or $M \models \mathrm{css}(N)$)

The following relations are definable and absolute:

- ϕ is in the syntactic form of a canonical Scott sentence
- $\phi = \mathrm{css}(M)$

Consistency

Proofs in $L_{\infty\omega}$:

- Predictable axiom set
- $\phi, \phi \to \psi \vdash \psi$
- $\{\phi_i : i \in I\} \vdash \bigwedge_{i \in I} \phi_i$
- $\phi_i \vdash \bigvee_{i \in I} \phi_i$

Proofs are now trees which are well-founded but possibly infinite.

$\phi \in L_{\infty\omega}$ is consistent if it does not prove $\neg\phi$.

Warning: folklore

Consistency, II

If $\phi \in L_{\omega_1\omega}$ is formally consistent, then it has a model.

This is not true for larger sentences:

- Let $\psi = \mathrm{css}(\omega_1, <)$, so ψ has no countable models.
- Let $L = \{<\} \cup \{c_n : n \in \omega\}$.
- Let $\phi = \psi \wedge (\forall x \bigvee_n x = c_n)$

Then ϕ is formally consistent, but ϕ has no models.

Fact: the property "ϕ is consistent" is absolute.

Potential Cardinality

Let $\Phi \in L_{\omega_1\omega}$. $\sigma \in L_{\infty\omega}$ is a potential canonical Scott sentence of Φ if:

1. σ has the syntactic form of a CSS
2. σ is formally consistent
3. σ formally proves Φ

Let $\mathrm{CSS}(\Phi)$ be the set of all these sentences. Let $\|\Phi\| = |\mathrm{CSS}(\Phi)|$.

Easy fact: $I(\Phi, \aleph_0) \leq I_{\infty\omega}(\Phi) \leq \|\Phi\|$.

Note: $I_{\infty\omega}(\Phi)$ is the number of models of Φ up to $\equiv_{\infty\omega}$

The Connection

If $f : \Phi \leq_B \Psi$, then f induces an injection from the countable Scott sentences of Φ to the countable Scott sentences of Ψ.

Theorem (Ulrich, R., Laskowski)

If $f : \Phi \leq_B \Psi$, then get an injection $\overline{f} : \mathrm{CSS}(\Phi) \to \mathrm{CSS}(\Psi)$.

Proof Idea:

- Fix $\tau \in \mathrm{CSS}(\Phi)$.
- $\overline{f}(\tau)$ is what f *would* take τ to, in some $\mathbb{V}[G]$ making τ countable.
- Schoenfield: "$\exists M \in \mathrm{Mod}_\omega(\Phi)\ (M \models \tau \wedge f(M) \models \sigma)$" is absolute
- If G_1 and G_2 are independent, then $\mathbb{V}[G_1] \cap \mathbb{V}[G_2] = \mathbb{V}\ldots$
- …so $\overline{f}(\tau) \in \mathbb{V}$ and $\overline{f}(\tau) \in \mathrm{CSS}(\Psi)$.

Some Easy Facts

Fact: If Φ is Borel, then $\|\Phi\| < \beth_{\omega_1}$

Proof Idea:

- Hjorth, Kechris, Louveau: If Φ is Π^0_α, then Φ is reducible to $\cong_\alpha$.
- $\| \cong_\alpha \| = \beth_{-1+\alpha+1}$, so $\|\Phi\| \leq \beth_{-1+\alpha+1}$.

Fact: If Φ is Borel complete, then $\|\Phi\| = \infty$

Proof Idea:

- (Folklore): all ordinals are back-and-forth inequivalent, so $\|\mathrm{LO}\| = \infty$.
- $\mathrm{LO} \leq_B \Phi$, so $\|\Phi\| = \infty$.

Some Excellent Questions

Hanf Number: Is it possible to get $\beth_{\omega_1} \leq \|\Phi\| < \infty$?

Unknown!

Is it possible for $\|\Phi\| = \infty$ when Φ is not Borel complete?

Yes!

Unknown if there are first-order examples

Is it possible for $\|\Phi\| < \beth_{\omega_1}$ when Φ is not Borel?

Yes! And there are first-order examples!

The last "yes!" answers a stubborn conjecture:

Can a first-order theory be neither Borel nor Borel complete?

A First Order Example

Let REF have language $L = \{E_n : n \in \omega\}$.

Axioms:

- Each E_n is an equivalence relation on the universe with 2^n classes.
- Each E_n-class splits into exactly two E_{n+1} classes.

REF is complete with quantifier elimination.
REF is superstable but not $\aleph_0$-stable.

REF Is Not Complicated

Despite not being Borel, REF is really nice:

- $I_{\infty\omega}(\mathrm{REF}) = \beth_2$:
  Idea: for all M, there is $N \subseteq M$ where $M \equiv_{\infty\omega} N$ and $|N| \leq \beth_1$.

- REF is grounded – for all $\Phi \in \mathrm{CSS}(\mathrm{REF})$, there is $M \models \Phi$ in $\mathbb{V}$.
  Idea:
  - Let $\mathbb{V}[G]$ collapse $|\Phi|$ to $\aleph_0$, let $N \models \Phi$ be its countable model.
  - Compute a bunch of invariants $\mathcal{I}(N)$ in $\mathbb{V}[G]$.
  - $\mathcal{I}(N) \in \mathbb{V}$, even though N is not.
  - Construct $M \models \Phi$ from $\mathcal{I}(N)$.

So: $\|\mathrm{REF}\| = \beth_2$, so $\cong_3 \not\leq_B \mathrm{REF}$, so REF is not Borel complete.

REF is Not Borel

REF has countable models of arbitrarily high Scott ranks.

Proof Sketch:

- Fix $A, B \models \mathrm{REF}$ countable where $A \equiv_\alpha B$ and $A \ncong B$.
- Construct models M_1 and M_2 where $M_1 \ncong M_2$ and $M_1 \equiv_{\alpha+1} M_2$.