Potential Cardinality, II

for Countable First-Order Theories

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A Reminder

A potential Scott sentence of some theory Φ is some $\phi \in L_{\infty\omega}$ where in some $\mathbb{V}[G]$, $\phi = \mathrm{css}(M)$ for some $M \models \Phi$.

$\mathrm{CSS}(\Phi)$ is the class of all potential Scott sentences.

$\|\Phi\| = |\mathrm{CSS}(\Phi)|$, which is possibly ∞.

Theorem (Ulrich, Rast, Laskowski)

If $\Phi \leq_B \Psi$, then $\|\Phi\| \leq \|\Psi\|$.

Some Excellent Questions

Hanf Number: Is there a Φ where $\beth_{\omega_1} \leq \|\Phi\| < \infty$?

Unknown!

Is it possible for $\|\Phi\| = \infty$ when Φ is not Borel complete?

Yes!

Unknown if there are first-order examples

Is it possible for $\|\Phi\| < \beth_{\omega_1}$ when Φ is not Borel?

Yes! And there are first-order examples!

The last "yes!" answers a stubborn conjecture:

Can a first-order theory be neither Borel nor Borel complete?

Abelian p-groups, I

Theorem (Friedman, Stanley)

Let Φ be the sentence describing abelian p-groups, for some prime p. Then Φ is not Borel and not Borel complete. Also, $\|\Phi\| = \infty$.

Sketch: Φ is not Borel (in fact $\|\Phi\| = \infty$)

- Define $F^\alpha(A)$ as the $\mathbb{F}_p$-dimension of $p^\alpha A / p^{\alpha+1} A$.
- Define $F^\infty(A)$ as the $\mathbb{Z}(p^\infty)$-dimension of $p^\infty A$.
  For both, use ∞ for all infinite dimensions.

- [Mackey/Kaplansky] $A \equiv_{\infty\omega} B$ iff $F^\alpha(A) = F^\alpha(B)$ for all $\alpha \leq \infty$

- If $|A|^+ \leq \alpha < \infty$, $F^\alpha(A) = 0$, but...
- ...one can construct A where $U^A_\alpha \neq 0$ [Kurosh]

- So $\infty = I_{\infty\omega}(\Phi) \leq \|\Phi\|$, so Φ is not Borel.

Note: this is slightly simpler, but very similar to the original FS argument

Abelian p-groups, II

Theorem (Friedman, Stanley)

Let Φ be the sentence describing abelian p-groups, for some prime p. Then Φ is not Borel and not Borel complete. Also, $\|\Phi\| = \infty$.

Sketch: $\cong_2 \not\leq_B \Phi$

- Suppose $f : \cong_2 \leq_B \Phi$.
- Then we get a "sufficiently definable" injection $\overline{f} : [\mathbb{R}]^{\aleph_0} \to [\omega_1]^{\aleph_0}$.
- [Friedman] no such map exists:
  - If G codes an ω-sequence of generic reals, then
  - The values $\overline{f}(G)(\alpha)$ are forced by $\emptyset$, so
  - $\overline{f}$ isn't injective in $\mathbb{V}[G]$, so
  - $\overline{f}$ isn't injective in $\mathbb{V}$ ※

So it is possible for Φ to be neither Borel nor Borel complete.
What about for a first-order theory?

Three First Order Examples

We worked with three complete first-order theories: REF, K, and TK.

REF is superstable, classifiable (depth 1), and not $\aleph_0$-stable.
$\|\mathrm{REF}\| = \beth_2$, so REF is not Borel complete, but REF is not Borel.

K is $\aleph_0$-stable and classifiable (depth 2).
$\|\mathrm{K}\| = \beth_2$, so K is not Borel complete, but K is not Borel.

TK is $\aleph_0$-stable and classifiable (depth 2).
TK is Borel complete, so $\|\mathrm{TK}\| = \infty$, but $I_{\infty\omega}(\mathrm{TK}) = \beth_2$.

REF is grounded; TK is not; groundedness of K is open.

Roadmap

Refining Equivalence Relations

REF is in the following language: $L = \{E_n : n \in \omega\}$. REF states:

1. Each E_n is an equivalence relation, all classes infinite
2. E_n has exactly 2^n classes
3. Each E_n class refines into exactly E_{n+1} classes

REF is superstable but not $\aleph_0$-stable (type counting).

In fact REF is super nice from a stability-theory perspective.

Refining Equivalence Relations, Overview

REF is the first known example of a first-order theory which is neither Borel nor Borel complete. We're going to show the following:

Niceness properties:

- Show $\cong_2 \leq_B \mathrm{REF}$ and $\beth_2 \leq I_{\infty\omega}(\mathrm{REF})$
- Show REF is grounded (every Scott sentence has a model), so...
- ... $\|\mathrm{REF}\| = \beth_2$, so ...
- $\cong_3 \not\leq_B \mathrm{REF}$, and REF is not Borel complete.

Non-niceness properties:

- REF admits countable models of large Scott rank, so ...
- ... REF is not Borel.

REF has Many Countable Models

We can embed "countable sets of reals" into $\mathrm{Mod}_\omega(\mathrm{REF})$.

Proof sketch:

- Pretend we have names from 2^n for each E_n class
- Then we have names from 2^ω for each E_∞ class
- Any *dense* $X \subset 2^\omega$ can be the set of E_∞ class we actually realize (say, realize them infinitely many times)

- Coding trick: we can realize certain E_∞ classes finitely many times, so that we still get this naming

So $\cong_2 \leq_B \mathrm{REF}$ and $I_{\infty\omega}(\mathrm{REF}) \geq \beth_2$

Aside: Properties of Things that Should Exist

For some theory T, fix $\phi \in \mathrm{CSS}(T)$.

Idea: even if ϕ has no models in $\mathbb{V}$, invariants of its "canonical model" can be computed in $\mathbb{V}$ anyway.

Sketch:

- Let $\mathbb{V}[G_1]$ and $\mathbb{V}[G_2]$ be independent and collapse $|\phi|$ to $\aleph_0$.
- Let M_i be the "unique" countable model of ϕ in $\mathbb{V}[G_i]$.
- Compute a set $\mathcal{I}_i$ which depends *only* on $M_i/\cong$
- Let $\mathbb{V}[G]$ contain $\mathbb{V}[G_1]$ and $\mathbb{V}[G_2]$.
- In $\mathbb{V}[G]$, $M_1 \cong M_2$ so $\mathcal{I}_1 = \mathcal{I}_2 =: \mathcal{I}$, and...
- $\ldots \mathcal{I} \in \mathbb{V}[G_1] \cap \mathbb{V}[G_2] = \mathbb{V}$.

REF is Grounded

Recall: ϕ is grounded if everything in $\mathrm{CSS}(\phi)$ has a model.

Theorem: Let $\phi \in \mathrm{CSS}(\mathrm{REF})$. Then ϕ has a model.
The Invariants of ϕ (or rather, $M \models \phi$ in $\mathbb{V}[G]$):

1. The tree of Scott sentences from naming specific E_n-classes in M,
2. The multiplicity at each node (1 or 2),
3. The set of branches through the tree which are actually realized, and
4. For each branch: the set of colors of elements yielding the branch

Tedious: $\phi = \psi$ iff they have the same invariants.
Concrete: Can construct a model of ϕ from its invariants.

REF is not Borel Complete

Theorem: $I_{\infty\omega}(\mathrm{REF}) = \beth_2$

Proof sketch:

- We already know $I_{\infty\omega}(\mathrm{REF}) \geq \beth_2$

- Let $M \models \mathrm{REF}$ be arbitrary.
- Let $N \subset M$ drop all but a countable subset of each E_∞ class
- $|N| \leq \beth_1$ and $M \equiv_{\infty\omega} N$.

- There are at most $\beth_2$ models of size $\beth_1$, up to $\equiv_{\infty\omega}$
- So $I_{\infty\omega}(\mathrm{REF}) \leq \beth_2$

Corollary: $\|\mathrm{REF}\| = \beth_2$

Corollary: REF is not Borel complete (in fact $\cong_3 \not\leq_B \mathrm{REF}$)

So Far, So Normal

What we know so far:

- REF is tame, from a stability-theory perspective
- REF is grounded
- $\|\mathrm{REF}\| = I_{\infty\omega}(\mathrm{REF})$, and both are a reasonable, small number
- REF is not Borel complete

Everything right now makes REF look very well-behaved.

Back-and-Forth Games, I

Let M and N be structures, $\overline{a} \in M^k$, $\overline{b} \in N^k$.
Define $(M, \overline{a}) \equiv_\alpha (N, \overline{b})$ by induction:

- $(M, \overline{a}) \equiv_0 (N, \overline{b})$ if for all atoms R, $M \models R(\overline{a})$ iff $N \models R(\overline{b})$
- $(M, \overline{a}) \equiv_\lambda (N, \overline{b})$ iff for all $\alpha < \lambda$, $(M, \overline{a}) \equiv_\alpha (N, \overline{b})$
- $(M, \overline{a}) \equiv_{\alpha+1} (N, \overline{b})$ iff
  $\forall c \in M \; \exists d \in N \; (M, \overline{a}c) \equiv_\alpha (N, \overline{b}d)$, and
  $\forall d \in N \; \exists c \in M \; (M, \overline{a}c) \equiv_\alpha (N, \overline{b}d)$

Easy: If M and N are countable, $M \cong N$ iff $M \equiv_{\omega_1} N$.
Fun Fact: $M \equiv N$ iff $M \equiv_\omega N$

Classical: $\cong_\Phi$ is Borel iff for some $\alpha < \omega_1$, $\equiv_\alpha$ is $\cong_\Phi$.

Back-and-Forth Games, II

Let M and N be structures, $\overline{a} \in M^k$, $\overline{b} \in N^k$.
Define the α-game for $(M, \overline{a})$ and $(N, \overline{b})$ as follows:

- On turn k, player I plays an ordinal α_k and an element of M or N
- Require $\alpha > \alpha_0 > \alpha_1 > \cdots$, so the game has finite length
- Player II responds with an element of N or M (respectively)
- At the end there is a tuple $\overline{c}$ from M and $\overline{d}$ from N
- Player II wins if $(M, \overline{ac}) \equiv_0 (N, \overline{bd})$

Induction: $(M, \overline{a}) \equiv_\alpha (N, \overline{b})$ iff player II has a winning strategy.

Bounded Branching Bubble Models

Theorem: REF is not Borel.
Sketch:

- By induction on α, construct $M, N \models \mathrm{REF}$ where $M \ncong N$, $M \equiv_\alpha N$.
- REF is complete and not $\aleph_0$-categorical, so $\alpha = 0$ works.
- Given $A \equiv_\alpha B$ and $A \ncong B$, and $X \subset 2^\omega$ dense, construct $M_X^{A,B}$ where the E_∞-class of $\eta \in 2^\omega$ is realized iff $\eta \in X$. Picture!
- If $Y \subset 2^\omega$ is dense, $M_X^{A,B} \equiv_{\alpha+1} M_Y^{A,B}$
- If $Y \neq X$, $M_X^{A,B} \ncong M_Y^{A,B}$

Note: limit case is similar, but slightly more complicated.

Wrapup on REF

Thus REF is an example of the following:

- A complete first order theory in a countable language, where
- The isomorphism relation is not Borel, and
- The isomorphism relation is not Borel complete

More importantly: potential cardinality gives a way to show the nonexistence of a Borel reduction, even when the underlying isomorphism relation is not Borel.

Side benefit: the proof was model-theoretic, rather than set-theoretic.

Note: after naming $\mathrm{acl}(\emptyset)$, the theory is Borel – in fact exactly $\cong_2$.

Roadmap

The Omega-stable Examples

We focus on two $\aleph_0$-stable theories K and TK.

Similarities:

- Both are $\aleph_0$-stable, classifiable, and have (eni)-depth 2.
- Both have non-Borel isomorphism relations.
- After naming constants for $\mathrm{acl}(\emptyset)$, the theories are identical.

Differences:

- $\|\mathrm{K}\| = \beth_2$, while
- TK is Borel complete.

- $\mathrm{Aut}(\mathrm{acl}(\emptyset))$ for K is $(2^\omega, +)$, while
- $\mathrm{Aut}(\mathrm{acl}(\emptyset))$ for TK is very complex.

Koerwien's Example

The theory K is in the language $L = \{U, C_n, V_n, S_n, \pi_n : n \in \omega\}$. K states:

- U and each of the V_n are infinite sorts; C_n is a sort of size two
- $\pi_n : V_n : U \times C_0 \times \cdots C_n$ is a surjection
- $S_n : V_n \to V_n$ is a successor function
- $\pi_n \circ S_n = \pi_n$

All the complexity is in deciding the dimension (color) of $\pi^{-1}(u, \overline{c})$
where $u \in U$ and $\overline{c} \in C_0 \times \cdots \times C_n$

K is $\aleph_0$-stable, classifiable, and has (eni)-depth two.

Theorem (Koerwien): $\cong_{\mathrm{K}}$ is not Borel.

Koerwien's Example, II

Fact: $\cong_2 \leq_B \mathrm{K}$ and $I_{\infty\omega}(\mathrm{K}) \geq \beth_2$.
Proof:

- Today's reals are ω^ω.
- Given an infinite $X \subset \omega^\omega$, construct $M_X \models \mathrm{K}$.

- Let $U_X = X$.
- For each $u \in U_x$ and each $n \in \omega, \ldots$
- Give $\pi^{-1}(u, \overline{c})$ dimension $u(n) + 1$ for all $\overline{c} \in C_0 \times \cdots \times C_{n-1}$.

- Easy to see $M_X \equiv_{\infty\omega} M_Y$ iff $X = Y$.

Koerwien's Example, III

Lemma: Let X be a recursively presented Polish space,
G be a compact abelian Polish group acting continuously on X,
$\mathcal{X} = \mathcal{P}_{\leq\aleph_0}(X)$, and $\mathcal{E}$ be the orbit equivalence relation of G on $\mathcal{X}$.
Then $\|(\mathcal{X}, \mathcal{E})\| \leq \beth_2$.

Sketch:

- Sufficient to show all $\phi \in \mathrm{CSS}(\mathcal{X}, \mathcal{E})$ are in $L_{\beth_1^+,\omega}$.
- To show that, sufficient to show $|S_n^\infty(\phi)| \leq \beth_1$ for all n.

- Use compactness to represent $\mathcal{E}$-classes as Scott sentences.
- Use abelianness to control the branching from S_n^α to S_n^∞.

Cor: $\|\mathrm{K}\| = \beth_2$, so $\cong_3 \not\leq_B \mathrm{K}$ and K is not Borel complete.

The Koerwien Tweak

The theory TK is in the language $L = \{U, C_n, V_n, S_n, \pi_n, p_n : n \in \omega\}$. TK states:

- U and each of the V_n are infinite sorts; C_n is a sort of size 2^n
- $\pi_n : V_n \to U \times C_n$ is a surjection
- $p_n : C_{n+1} \to C_n$ is a two-to-one surjection
- $S_n : V_n \to V_n$ is a successor function
- $\pi_n \circ S_n = \pi_n$

TK is $\aleph_0$-stable, classifiable, and has (eni)-depth two.

The only apparent difference between K and TK is $\mathrm{Aut}(\mathrm{acl}(\emptyset))$; here it's nonabelian and complicated.

After naming $\mathrm{acl}(\emptyset)$, $\mathrm{K} \sim_B \mathrm{TK} \sim_B \cong_2$.

The Koerwien Tweak, II

Theorem: TK is Borel complete.

Sketch:

- Enough to code graphs on ω into models of TK.
- For each pair (i, j) from ω, get lots of corresponding nodes u where:
  - If $i = j$, then u has "color" 1.
  - If $i \neq j$ but they're connected, then u has "color" 2.
  - If $i \neq j$ and they're not connected, then u has "color" 3.
- Let $\{D_i : i \in \omega\}$ be countable, disjoint, dense subsets of 2^ω.
- The nodes are indexed by pairs (η, τ) from $D = \bigcup_i D_i$.
- $u_{\eta,\tau}$ corresponds to (i, j) iff $\eta \in D_i$ and $\tau \in D_j$.
- The nodes finite dimension on $\sigma \in 2^n$ iff $\sigma \subset \eta \cap \tau$.
- Claim: $\forall \sigma \in S_\infty$, $\exists g \in \mathrm{Aut}(\mathrm{acl}(\emptyset))$ where $g(D_i) = D_{\sigma(i)}$ as sets.

Dividing Lines?

Question: Does the Borel complexity of T correspond to anything model-theoretic about T?

There are some positive results around:

- In o-minimal theories, either T is Borel complete or $T \leq_B \cong_2$, depending on nonsimple types
- In $\aleph_0$-stable theories, eni-depth gives a lower bound for complexity: If $e(T) \geq 2 + \alpha$, then $\cong_\alpha \leq_B T$

But a lot of poorly understood behavior:

- Boring automorphism groups can deny complexity (K versus TK)...
- But difficult groups are not enough to guarantee it (REF versus TK)

So if there are dividing lines, it's not clear where they are, or what they divide.