# Potential Cardinality, I

## for Countable First-Order Theories

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# The Main Idea

The Goal: Understand the countable models of a theory $\Phi$

Chosen framework: if $\Phi \leq_B \Psi$ then the countable models of $\Phi$ are "more tame" than the countable models of $\Psi$.

Relatively easy: show $\Phi \leq_B \Psi$;
Relatively hard: show $\Phi \not\leq_B \Psi$

**Theorem (Ulrich, R., Laskowski)**

If $\Phi \leq_B \Psi$ then $\|\Phi\| \leq \|\Psi\|$.

# Roadmap

1. Borel Reductions
2. Back-and-Forth Equivalence, Scott Sentences, and Potential Cardinality
3. Computations and Consequences

# Motivation?

Why study Borel reductions?

Comparing the number of models is pretty coarse. Consider:

1. Countable sets of $\mathbb{Q}$-vector spaces
2. Graphs

These both have $\beth_1$ countable models, but
Borel reductions can easily show the former is much smaller than the latter.

Counterexamples to Vaught's conjecture are pretty weird;
Borel reductions give a nice way to make this formal (even given $\mathrm{CH}$).

# Borel Reductions

Fix $\Phi, \Psi \in L_{\omega_1\omega}$.

$\mathrm{Mod}_\omega(\Phi)$ and $\mathrm{Mod}_\omega(\Psi)$ are Polish spaces under the formula topology.

$f : \mathrm{Mod}_\omega(\Phi) \to \mathrm{Mod}_\omega(\Psi)$ is a Borel reduction if:

1. For all $M, N \models \Phi$, $M \cong N$ iff $f(M) \cong f(N)$
2. For any $\psi \in L_{\omega_1\omega}$ (with parameters from $\omega$)
   there is a $\phi \in L_{\omega_1\omega}$ (with parameters from $\omega$)
   where $f^{-1}(\mathrm{Mod}_\omega(\Psi \wedge \psi)) = \mathrm{Mod}_\omega(\Phi \wedge \phi)$

   (preimages of Borel sets are Borel)

Say $\Phi \leq_B \Psi$.

# A Real Example

Let $\Phi$ be "linear orders" and $\Psi$ be "real closed fields." Then $\Phi \leq_B \Psi$.

Proof outline:

- Fix a linear order $(I, <)$
- Pick a sequence $(a_i : i \in I)$ from the monster RCF where $1 \ll a_i$ for all $i$, and if $i < j$, then $a_i \ll a_j$.
- Let $M_I$ be prime over $\{a_i : i \in I\}$.
- $f$ is "obviously Borel"
- $(I, <) \cong (J, <)$ iff $M_I \cong M_J$.

# Establishing Some Benchmarks

Borel reducibility is inherently relative; it's hard to gauge complexity of (the countable models of) a sentence on its own.

One fix is to establish some benchmarks.
The two most important (for us) are:

- Being Borel – a tameness condition which isn't too degenerate
  Can stratify this into (e.g.) $\Pi^0_\alpha$ for each $\alpha < \omega_1$
- Being Borel complete – being maximally complicated

# Borel Isomorphism Relations

Fix $\Phi \in L_{\omega_1\omega}$. The following are equivalent:

1. Isomorphism for $\Phi$ is Borel (as a subset of $\mathrm{Mod}_\omega(\Phi)^2$)
2. There is a countable bound on the Scott ranks of all countable models
3. There is an $\alpha < \omega_1$ where $\equiv_\alpha$ implies $\cong$ for countable models of $\Phi$
4. There is a countable bound on the Scott ranks of all models of $\Phi$
5. There is an $\alpha < \omega_1$ where $\equiv_\alpha$ implies $\equiv_{\infty\omega}$ for all models of $\Phi$.

Fact: if $\Phi$ is Borel and $\Psi \leq_B \Phi$, then $\Psi$ is Borel.

# Borel Complete Isomorphism Relations

Fix $\Phi \in L_{\omega_1\omega}$. $\Phi$ is Borel complete if, for all $\Psi$, $\Psi \leq_B \Phi$.

**Theorem (Friedman, Stanley)**

Lots of classes are Borel complete:

- Graphs
- Trees
- Linear orders
- Groups
- Fields
- ...

Fact: If $\Phi$ is Borel complete, then $\Phi$ is not Borel.

# A Serious Question

It's somewhat clear how to show that $\Phi \leq_B \Psi$.

How is it possible to show that $\Phi \not\leq_B \Psi$?

Partial answer: there are some techniques, but they only apply when $\Phi$ or $\Psi$ is Borel (and low in the hierarchy).

Very little is known when you can't assume Borel.

# Roadmap, II

1. Borel Reductions
2. Back-and-Forth Equivalence, Scott Sentences, and Potential Cardinality
3. Computations and Consequences

# Back-and-Forth Equivalence

Let $M$ and $N$ be $L$-structures. $\mathcal{F} : M \to N$ is a back-and-forth system if:

1. $\mathcal{F}$ is a nonempty set of partial functions $M \to N$
2. All $f \in \mathcal{F}$ preserve $L$-atoms and their negations
3. For all $f \in \mathcal{F}$, all $m \in M$, and all $n \in N$, there is a $g \in \mathcal{F}$ where $m \in \mathrm{dom}(g)$, $n \in \mathrm{im}(g)$, and $f \subset g$

Say $M \equiv_{\infty\omega} N$ if there is such an $\mathcal{F}$.

If $M \cong N$ then $M \equiv_{\infty\omega} N$.
If $M$ and $N$ are countable and $M \equiv_{\infty\omega} N$, then $M \cong N$.

# Back-and-Forth Equivalence, II

$M \equiv_{\infty\omega} N$ means they are the same from an "intrinsic perspective."

More precisely, the following are equivalent:

- $M \equiv_{\infty\omega} N$
- For every $\phi \in L_{\infty\omega}$, $M \models \phi$ iff $N \models \phi$
- In some $\mathbb{V}[G]$, $M \cong N$
- In every $\mathbb{V}[G]$ making $M$ and $N$ countable, $M \cong N$

The relation "$M \equiv_{\infty\omega} N$" is absolute.

Canonical Scott sentences form a canonical invariant of each $\equiv_{\infty\omega}$-class. Given an $L$-structure $M$, a tuple $\overline{a}$, and an ordinal $\alpha$, define $\phi_\alpha^{\overline{a}}(\overline{x})$ as follows:

$$\phi_0^{\overline{a}}(\overline{x}) \text{ is } \mathrm{qftp}(\overline{a})$$

$$\phi_\lambda^{\overline{a}}(\overline{x}) \text{ is } \bigwedge_{\beta<\lambda} \phi_\beta^{\overline{a}}(\overline{x}) \text{ for limit } \lambda$$

$$\phi_{\beta+1}^{\overline{a}}(\overline{x}) \text{ is } \phi_\beta^{\overline{a}}(\overline{x}) \wedge \left(\forall y \bigvee_{b\in M} \phi_\beta^{\overline{a}b}(\overline{x}y)\right) \wedge \bigwedge_{b\in M} \exists y \phi_\beta^{\overline{a}b}(\overline{x}y)$$

For some minimal $\alpha^*$, for all $\overline{a} \in M$, $\phi_{\alpha^*}^{\overline{a}}(\overline{x})$ implies $\phi_{\alpha^*+1}^{\overline{a}}(\overline{x})$.

Define $\mathrm{css}(M)$ as $\phi_{\alpha^*}^{\emptyset} \wedge \bigwedge_{\overline{a}\in M} \forall \overline{x} \phi_{\alpha^*}^{\overline{a}}(\overline{x}) \rightarrow \phi_{\alpha^*+1}^{\overline{a}}(\overline{x})$

# Canonical Scott Sentences, II

For all $M$, $N$, the following are equivalent:

1. $M \equiv_{\infty\omega} N$
2. $\mathrm{css}(M) = \mathrm{css}(N)$
3. $N \models \mathrm{css}(M)$ (and/or $M \models \mathrm{css}(N)$)

Also, if $|M| \leq \lambda$, then $\mathrm{css}(M) \in L_{\lambda^+\omega}$.

Also, the relation "$\phi = \mathrm{css}(M)$" is absolute.

Also also, the property "$\phi$ is in the form of a canonical Scott sentence" is definable and absolute.

# Consistency

Proofs in $L_{\infty\omega}$:

- Predictable axiom set
- $\phi, \phi \to \psi \vdash \psi$
- $\{\phi_i : i \in I\} \vdash \bigwedge_{i \in I} \phi_i$

Proofs are now trees which are well-founded but possibly infinite.

$\phi \in L_{\infty\omega}$ is consistent if it does not prove $\neg\phi$.

Warning: folklore

# Consistency, II

If $\phi \in L_{\omega_1\omega}$ is formally consistent, then it has a model.

This is not true for larger sentences:

- Let $\psi = \mathrm{css}(\omega_1, <)$, so $\psi$ has no countable models.
- Let $L = \{<\} \cup \{c_n : n \in \omega\}$.
- Let $\phi = \psi \wedge (\forall x \bigvee_n x = c_n)$

Then $\phi$ is formally consistent, but $\phi$ has no models.

Fact: the property "$\phi$ is consistent" is absolute.

# Potential Cardinality

Let $\Phi \in L_{\omega_1\omega}$. $\sigma \in L_{\infty\omega}$ is a potential canonical Scott sentence of $\Phi$ if:

1. $\sigma$ has the syntactic form of a CSS
2. $\sigma$ is formally consistent
3. $\sigma$ proves $\Phi$

Let $\mathrm{CSS}(\Phi)$ be the set of all these sentences. Let $\|\Phi\| = |\mathrm{CSS}(\Phi)|$.

Easy fact: $I(\Phi, \aleph_0) \leq I_{\infty\omega}(\Phi) \leq \|\Phi\|$.

Note: $I_{\infty\omega}(\Phi)$ is the number of models of $\Phi$ up to $\equiv_{\infty\omega}$

# The Connection

If $f : \Phi \leq_B \Psi$, then $f$ induces an injection from the countable Scott sentences of $\Phi$ to the countable Scott sentences of $\Psi$.

**Theorem (Ulrich, R., Laskowski)**

If $f : \Phi \leq_B \Psi$, then get an injection $\overline{f} : \mathrm{CSS}(\Phi) \to \mathrm{CSS}(\Psi)$.

Proof Idea:

- Fix $\tau \in \mathrm{CSS}(\Phi)$.
- $\overline{f}(\tau)$ is what $f$ *would* take $\tau$ to, in some $\mathbb{V}[G]$ making $\tau$ countable.
- Schoenfield: "$\exists M \in \mathrm{Mod}_\omega(\Phi)\ (M \models \tau \wedge f(M) \models \sigma)$" is absolute
- General fact: If $G_1$ and $G_2$ are independent, then $\mathbb{V}[G_1] \cap \mathbb{V}[G_2] = \mathbb{V} \ldots$
- $\ldots$ so $\overline{f}(\tau) \in \mathbb{V}$ and $\overline{f}(\tau) \in \mathrm{CSS}(\Psi)$.

# Roadmap

# A Few Examples

- If $T$ is $\aleph_0$-categorical, $\|T\| = 1$.
- If $T$ is the theory of algebraically closed fields, $\|T\| = \aleph_0$:
  Coded by the transcendence degree: 0, 1, 2, ... or "infinite."
- If $T = (\mathbb{Q}, <, c_q)_{q \in \mathbb{Q}}$, then $\|T\| = \beth_2$.
  Models are coded by which 1-types they realize, and how.

All these examples are grounded – every potential Scott sentence has a model. Weirder examples won't have this property.

# Being Borel

FS/HKL: $\Phi$ is Borel iff $\Phi \leq_B \cong_\alpha$ for some $\alpha < \omega_1$.

**Corollary**

If $\Phi$ is Borel, $\|\Phi\| < \beth_{\omega_1}$.

Proof Sketch:

- Define the *jump* of $\Psi$, $J(\Psi)$, to code "multisets of models of $\Psi$."
- Define the *limit jump* of $\Psi$ at limit ordinals $\lambda$ to be $\sqcup_{\alpha<\lambda} J^\alpha(\Psi)$.

- Easy: $J^\alpha(\cong_\beta) \sim_B \cong_{\beta+\alpha}$.
- Easy: $\| \cong_0 \| = \beth_0$
- Induction: $\|J^\alpha(\Psi)\| = \beth_{-1+\alpha+1}(\|\Psi\|)$

- If $\Phi \leq_B \cong_\alpha$, $\|\Phi\| \leq \beth_{-1+\alpha+1} < \beth_{\omega_1}$.

# Being Borel Complete

**Proposition**

If $\Phi$ is Borel complete, $\|\Phi\| = \infty$.

Proof Sketch:

- If $\Phi$ is Borel complete, $\mathrm{LO} \leq_B \Phi$, so $\|\mathrm{LO}\| \leq \|\Phi\|$.
- Folklore: all ordinals are back-and-forth inequivalent, so
- $\infty = I_{\infty\omega}(\mathrm{LO}) \leq \|\mathrm{LO}\| \leq \|\Phi\|$

# Some Excellent Questions

Hanf Number: Is it possible to get $\beth_{\omega_1} \leq \|\Phi\| < \infty$?
    Unknown!

Is it possible for $\|\Phi\| = \infty$ when $\Phi$ is not Borel complete?
    Yes!
    Unknown if there are first-order examples

Is it possible for $\|\Phi\| < \beth_{\omega_1}$ when $\Phi$ is not Borel?
    Yes! And there are first-order examples!

The last "yes!" answers a stubborn conjecture:
    Can a first-order theory be neither Borel nor Borel complete?