

# Potential Cardinality

## for Countable First-Order Theories

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# The Main Idea

**The Goal:** Understand the countable models of a theory  $\Phi$

Chosen framework: if  $\Phi \leq_B \Psi$  then the countable models of  $\Phi$  are “more tame” than the countable models of  $\Psi$ .

Relatively **easy**: show  $\Phi \leq_B \Psi$ ;

Relatively **hard**: show  $\Phi \not\leq_B \Psi$

**Theorem (Ulrich, R., Laskowski)**

If  $\Phi \leq_B \Psi$  then  $\|\Phi\| \leq \|\Psi\|$ .

# Roadmap

- 1 Borel Reductions
- 2 Back-and-Forth Equivalence, Scott Sentences, and Potential Cardinality
- 3 Connections
- 4 Extended Examples

# Motivation?

Why study Borel reductions?

Comparing the number of models is pretty coarse. Consider:

- 1 Countable sets of  $\mathbb{Q}$ -vector spaces
- 2 Graphs

These both have  $\beth_1$  countable models, but

Borel reductions can easily show the former is **much smaller** than the latter.

Counterexamples to Vaught's conjecture are **pretty weird**;

Borel reductions give a nice way to make this formal (even given CH).

# Borel Reductions

Fix  $\Phi, \Psi \in L_{\omega_1\omega}$ .

$\text{Mod}_\omega(\Phi)$  and  $\text{Mod}_\omega(\Psi)$  are Polish spaces under the formula topology.

$f : \text{Mod}_\omega(\Phi) \rightarrow \text{Mod}_\omega(\Psi)$  is a Borel reduction if:

- ① For all  $M, N \models \Phi$ ,  $M \cong N$  iff  $f(M) \cong f(N)$
- ② For any  $\psi \in L_{\omega_1\omega}$  (with parameters from  $\omega$ )  
there is a  $\phi \in L_{\omega_1\omega}$  (with parameters from  $\omega$ )  
where  $f^{-1}(\text{Mod}_\omega(\Psi \wedge \psi)) = \text{Mod}_\omega(\Phi \wedge \phi)$

(preimages of Borel sets are Borel)

Say  $\Phi \leq_B \Psi$ .

# A Real Example

Let  $\Phi$  be “linear orders” and  $\Psi$  be “real closed fields.” Then  $\Phi \leq_B \Psi$ .

Proof outline:

- Fix a linear order  $(I, <)$
- Pick a sequence  $(a_i : i \in I)$  from the monster RCF where  $1 \ll a_i$  for all  $i$ , and if  $i < j$ , then  $a_i \ll a_j$ .
- Let  $M_I$  be prime over  $\{a_i : i \in I\}$ .
- $f$  is “obviously Borel”
- $(I, <) \cong (J, <)$  iff  $M_I \cong M_J$ .

# Establishing Some Benchmarks

Borel reducibility is inherently **relative**; it's hard to gauge complexity of (the countable models of) a sentence on its own.

One fix is to establish some **benchmarks**.

The two most important (for us) are:

- Being **Borel** – a tameness condition which isn't too degenerate
- Being **Borel complete** – being maximally complicated

# Borel Isomorphism Relations

Fix  $\Phi \in L_{\omega_1\omega}$ . The following are equivalent:

- 1 Isomorphism for  $\Phi$  is Borel (as a subset of  $\text{Mod}_\omega(\Phi)^2$ )
- 2 There is a countable bound on the Scott ranks of all **countable** models
- 3 There is an  $\alpha < \omega_1$  where  $\equiv_\alpha$  implies  $\cong$  for **countable** models of  $\Phi$
- 4 There is a countable bound on the Scott ranks of **all** models of  $\Phi$
- 5 There is an  $\alpha < \omega_1$  where  $\equiv_\alpha$  implies  $\equiv_{\infty\omega}$  for **all** models of  $\Phi$ .

**Fact:** if  $\Phi$  is Borel and  $\Psi \leq_B \Phi$ , then  $\Psi$  is Borel.



# Borel Complete Isomorphism Relations

Fix  $\Phi \in L_{\omega_1\omega}$ .  $\Phi$  is **Borel complete** if, for all  $\Psi$ ,  $\Psi \leq_B \Phi$ .

## Theorem (Friedman, Stanley)

Lots of classes are Borel complete:

- Graphs
- Trees
- Linear orders
- Groups
- Fields
- ...

**Fact:** If  $\Phi$  is Borel complete, then  $\Phi$  is not Borel.

# A Serious Question

It's somewhat clear how to show that  $\Phi \leq_B \Psi$ .

How is it possible to show that  $\Phi \not\leq_B \Psi$ ?

**Partial answer:** there are some techniques, but they only apply when  $\Phi$  or  $\Psi$  is Borel (and low in the hierarchy).

**Very little is known** when you can't assume Borel.

# Roadmap, II

- 1 Borel Reductions
- 2 Back-and-Forth Equivalence, Scott Sentences, and Potential Cardinality
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# Back-and-Forth Equivalence

Let  $M$  and  $N$  be  $L$ -structures.  $\mathcal{F} : M \rightarrow N$  is a **back-and-forth system** if:

- ①  $\mathcal{F}$  is a nonempty set of partial functions  $M \rightarrow N$
- ② All  $f \in \mathcal{F}$  preserve  $L$ -atoms and their negations
- ③ For all  $f \in \mathcal{F}$ , all  $m \in M$ , and all  $n \in N$ ,  
there is a  $g \in \mathcal{F}$  where  $m \in \text{dom}(g)$ ,  $n \in \text{im}(g)$ , and  $f \subset g$

Say  $M \equiv_{\infty\omega} N$  if there is such an  $\mathcal{F}$ .

If  $M \cong N$  then  $M \equiv_{\infty\omega} N$ .

If  $M$  and  $N$  are **countable** and  $M \equiv_{\infty\omega} N$ , then  $M \cong N$ .

# Back-and-Forth Equivalence, II

$M \equiv_{\infty\omega} N$  means they are the same from an “intrinsic perspective.”

More precisely, the following are equivalent:

- $M \equiv_{\infty\omega} N$
- For every  $\phi \in L_{\infty\omega}$ ,  $M \models \phi$  iff  $N \models \phi$
- In some  $\mathbb{V}[G]$ ,  $M \cong N$
- In every  $\mathbb{V}[G]$  making  $M$  and  $N$  countable,  $M \cong N$

The relation “ $M \equiv_{\infty\omega} N$ ” is absolute.

# Canonical Scott Sentences

Canonical Scott sentences form a **canonical invariant** of each  $\equiv_{\infty\omega}$ -class. Given an  $L$ -structure  $M$ , a tuple  $\bar{a}$ , and an ordinal  $\alpha$ , define  $\phi_{\alpha}^{\bar{a}}(\bar{x})$  as follows:

$$\phi_0^{\bar{a}}(\bar{x}) \text{ is } \text{qftp}(\bar{a})$$

$$\phi_{\lambda}^{\bar{a}}(\bar{x}) \text{ is } \bigwedge_{\beta < \lambda} \phi_{\beta}^{\bar{a}}(\bar{x}) \text{ for limit } \lambda$$

$$\phi_{\beta+1}^{\bar{a}}(\bar{x}) \text{ is } \phi_{\beta}^{\bar{a}}(\bar{x}) \wedge \left( \forall y \bigvee_{b \in M} \phi_{\beta}^{\bar{a}b}(\bar{x}y) \right) \wedge \bigwedge_{b \in M} \exists y \phi_{\beta}^{\bar{a}b}(\bar{x}y)$$

For some minimal  $\alpha^*$ , for all  $\bar{a} \in M$ ,  $\phi_{\alpha^*}^{\bar{a}}(\bar{x})$  implies  $\phi_{\alpha^*+1}^{\bar{a}}(\bar{x})$ .

Define  $\text{css}(M)$  as  $\phi_{\alpha^*}^{\emptyset} \wedge \bigwedge_{\bar{a} \in M} \forall \bar{x} \phi_{\alpha^*}^{\bar{a}}(\bar{x}) \rightarrow \phi_{\alpha^*+1}^{\bar{a}}(\bar{x})$

# Canonical Scott Sentences, II

For all  $M, N$ , the following are equivalent:

- 1  $M \equiv_{\infty\omega} N$
- 2  $\text{css}(M) = \text{css}(N)$
- 3  $N \models \text{css}(M)$  (and/or  $M \models \text{css}(N)$ )

Also, if  $|M| = \lambda$ , then  $\text{css}(M) \in L_{\lambda+\omega}$ .

Also, the relation “ $\phi = \text{css}(M)$ ” is absolute.

Also also, the property “ $\phi$  is in the form of a canonical Scott sentence” is definable and absolute.

# Consistency

Proofs in  $L_{\infty\omega}$ :

- Predictable axiom set
- $\phi, \phi \rightarrow \psi \vdash \psi$
- $\{\phi_i : i \in I\} \vdash \bigwedge_{i \in I} \phi_i$

Proofs are now **trees** which are well-founded but possibly infinite.

$\phi \in L_{\infty\omega}$  is **consistent** if it does not prove  $\neg\phi$ .

**Warning:** folklore



# Consistency, II

If  $\phi \in L_{\omega_1\omega}$  is **formally consistent**, then it has a model.

This is not true for larger sentences:

- Let  $\psi = \text{css}(\omega_1, <)$ , so  $\psi$  has no countable models.
- Let  $L = \{<\} \cup \{c_n : n \in \omega\}$ .
- Let  $\phi = \psi \wedge (\forall x \bigvee_n x = c_n)$

Then  $\phi$  is **formally consistent**, but  $\phi$  has **no models**.

**Fact:** the property “ $\phi$  is consistent” is absolute.

# Potential Cardinality

Let  $\Phi \in L_{\omega_1\omega}$ .  $\sigma \in L_{\infty\omega}$  is a **potential canonical Scott sentence** of  $\Phi$  if:

- 1  $\sigma$  has the syntactic form of a CSS
- 2  $\sigma$  is formally consistent
- 3  $\sigma$  proves  $\Phi$

Let  $\text{CSS}(\Phi)$  be the set of all these sentences. Let  $\|\Phi\| = |\text{CSS}(\Phi)|$ .

**Easy fact:**  $I(\Phi, \aleph_0) \leq I_{\infty\omega}(\Phi) \leq \|\Phi\|$ .

**Note:**  $I_{\infty\omega}(\Phi)$  is the number of models of  $\Phi$  up to  $\equiv_{\infty\omega}$

# A Few Examples

- If  $T$  is  $\aleph_0$ -categorical,  $\|T\| = 1$ .
- If  $T$  is the theory of algebraically closed fields,  $\|T\| = \aleph_0$ :  
Coded by the transcendence degree: 0, 1, 2, ... or “infinite.”
- If  $T = (\mathbb{Q}, <, c_q)_{q \in \mathbb{Q}}$ , then  $\|T\| = \beth_2$ .  
Models are coded by which 1-types they realize, and how.

All these examples are **grounded** – every potential Scott sentence has a model. **Weirder examples won't have this property.**

# Roadmap

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# The Connection

If  $f : \Phi \leq_B \Psi$ , then  $f$  induces an injection from the countable Scott sentences of  $\Phi$  to the countable Scott sentences of  $\Psi$ .

## Theorem (Ulrich, R., Laskowski)

If  $f : \Phi \leq_B \Psi$ , then get an injection  $\bar{f} : \text{CSS}(\Phi) \rightarrow \text{CSS}(\Psi)$ .

### Proof Idea:

- Fix  $\tau \in \text{CSS}(\Phi)$ .
- $\bar{f}(\tau)$  is what  $f$  *would* take  $\tau$  to, in some  $\mathbb{V}[G]$  making  $\tau$  countable.
- **Schoenfield**: “ $\exists M \in \text{Mod}_\omega(\Phi) (M \models \tau \wedge f(M) \models \sigma)$ ” is absolute
- If  $G_1$  and  $G_2$  are independent, then  $\mathbb{V}[G_1] \cap \mathbb{V}[G_2] = \mathbb{V} \dots$
- ... so  $\bar{f}(\tau) \in \mathbb{V}$  and  $\bar{f}(\tau) \in \text{CSS}(\Psi)$ .

# Some Easy Facts

**Fact:** If  $\Phi$  is Borel, then  $\|\Phi\| < \beth_{\omega_1}$

**Proof Idea:**

- (Hjorth): If  $\Phi$  is  $\Pi^0_\alpha$ , then  $\Phi$  is reducible to  $\cong_\alpha$ .
- $\|\cong_\alpha\| = \beth_{-1+\alpha+1}$ , so  $\|\Phi\| \leq \beth_{-1+\alpha+1}$ .

**Fact:** If  $\Phi$  is Borel complete, then  $\|\Phi\| = \infty$

**Proof Idea:**

- (Folklore): all ordinals are back-and-forth inequivalent, so  $\|\text{LO}\| = \infty$ .
- $\text{LO} \leq_B \Phi$ , so  $\|\Phi\| = \infty$ .

# Some Excellent Questions

**Hanf Number:** Is it possible to get  $\beth_{\omega_1} \leq \|\Phi\| < \infty$ ?

Unknown!

Is it possible for  $\|\Phi\| = \infty$  when  $\Phi$  is not Borel complete?

Yes!

Unknown if there are first-order examples

Is it possible for  $\|\Phi\| < \beth_{\omega_1}$  when  $\Phi$  is not Borel?

Yes! And there are first-order examples!

The last “yes!” answers a stubborn conjecture:

Can a first-order theory be neither Borel nor Borel complete?

# One Answer

## Theorem (Friedman, Stanley)

Let  $\Phi$  be the sentence describing abelian  $p$ -groups, for some prime  $p$ . Then  $\Phi$  is not Borel and not Borel complete. Also,  $\|\Phi\| = \infty$ .

## Proof Sketch

- Can construct  $p$ -groups of arbitrary (ordinal) Ulm height, so  $\|\Phi\| = \infty$ , so  $\Phi$  is not Borel.
- Can't embed countable sets of reals into  $\Phi$ :  
Suitably generic sets of reals go to the same group, so **injectivity fails**.

So it is possible for  $\Phi$  to be neither Borel nor Borel complete.  
What about for a first-order theory?



# Three First Order Examples

We worked with three complete first-order theories: REF, K, and TK.

REF is superstable, classifiable (depth 1), and not  $\aleph_0$ -stable.

$\| \text{REF} \| = \beth_2$ , so REF is not Borel complete, but REF is not Borel.

K is  $\aleph_0$ -stable and classifiable (depth 2).

$\| K \| = \beth_2$ , so K is not Borel complete, but K is not Borel.

TK is  $\aleph_0$ -stable and classifiable (depth 2).

TK is Borel complete, so  $\| \text{TK} \| = \infty$ , but  $I_{\infty\omega}(\text{TK}) = \beth_2$ .

REF is grounded; TK is not; groundedness of K is open.

# Refining Equivalence Relations

REF is in the following language:  $L = \{E_n : n \in \omega\}$ . REF states:

- 1 Each  $E_n$  is an equivalence relation, all classes infinite
- 2  $E_n$  has exactly  $2^n$  classes
- 3 Each  $E_n$  class refines into exactly  $E_{n+1}$  classes

REF is superstable but not  $\aleph_0$ -stable (type counting).

In fact REF is **super nice** from a stability-theory perspective.

# REF has Many Countable Models

We can embed “countable sets of reals” into  $\text{Mod}_\omega(\text{REF})$ .

Proof sketch:

- Pretend we have names from  $2^n$  for each  $E_n$  class
- Then we have names from  $2^\omega$  for each  $E_\infty$  class
- Any *dense*  $X \subset 2^\omega$  can be the set of  $E_\infty$  class we actually realize (say, realize them infinitely many times)
- Coding trick: we can realize certain  $E_\infty$  classes finitely many times, so that we still get this naming

So  $\cong_{2 \leq B} \text{REF}$  and  $I_{\infty\omega}(\text{REF}) \geq \beth_2$

# REF is Grounded

Recall:  $\Phi$  is **grounded** if everything in  $\text{CSS}(\Phi)$  has a model.

**Theorem:** Let  $\phi \in \text{CSS}(\text{REF})$ . Then  $\phi$  has a model.

**Proof sketch:**

- Let  $\mathbb{V}[G]$  think  $\phi$  is countable, so it has a model
- The countable model  $M$  of  $\phi$  is *unique* up to isomorphism
- Compute a bunch of invariants of  $M$  in  $\mathbb{V}[G]$
- Even if  $M \notin \mathbb{V}$ , all the invariants are in  $\mathbb{V}$
- In  $\mathbb{V}$ , build a model  $N \models \text{REF}$
- In  $\mathbb{V}[G]$ , show  $M \equiv_{\infty\omega} N$ , so that  $N \models \phi$  in  $\mathbb{V}$

**Note:** the invariants are essentially a tree of Scott sentences extending  $\phi$ , in a larger language, plus some related trees

# REF is not Borel Complete

**Theorem:**  $I_{\infty\omega}(\text{REF}) = \beth_2$

**Proof sketch:**

- We already know  $I_{\infty\omega}(\text{REF}) \geq \beth_2$
- Let  $M \models \text{REF}$  be arbitrary.
- Let  $N \subset M$  drop all but a countable subset of each  $E_\infty$  class
- $|N| \leq \beth_1$  and  $M \equiv_{\infty\omega} N$ .
- There are at most  $\beth_2$  models of size  $\beth_1$ , up to  $\equiv_{\infty\omega}$
- So  $I_{\infty\omega}(\text{REF}) \leq \beth_2$

**Corollary:**  $\|\text{REF}\| = \beth_2$

**Corollary:** REF is not Borel complete

# So Far, So Normal

What we know so far:

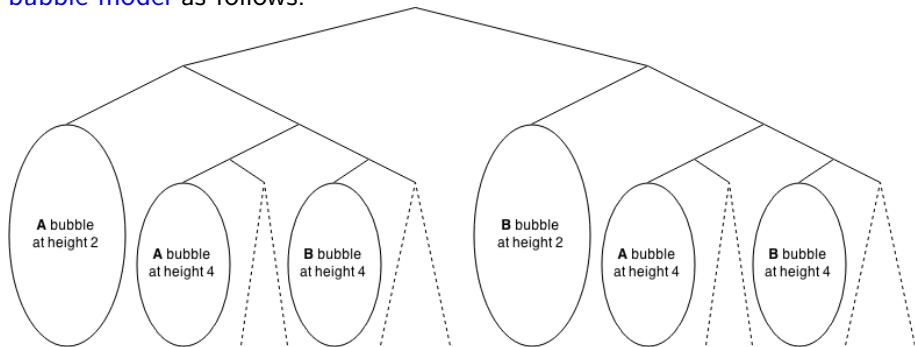
- REF is **tame**, from a stability-theory perspective
- REF is **grounded**
- $\|\text{REF}\| = I_{\infty\omega}(\text{REF})$ , and both are a reasonable, small number
- REF is not Borel complete

Everything right now makes REF look **very well-behaved**.

# REF is Not Borel

REF has countable models of arbitrarily high Scott ranks.

**The Construction:** Fix  $A, B \models \text{REF}$  countable where  $A \equiv_\alpha B$  and  $A \not\equiv B$ . Fix  $X \subset 2^\omega$  countable and dense. Construct  $M_X$  as a **branching balanced bubble model** as follows:



Realize the  $E_\infty$ -class of  $\eta \in 2^\omega$  iff  $\eta \in X$ .

If  $X \neq Y$ , then  $M_X \equiv_{\alpha+1} M_Y$  but  $M_X \not\equiv M_Y$ .

# Wrapup on REF

Thus REF is an example of the following:

- A complete first order theory in a countable language, where
- The isomorphism relation is not Borel, and
- The isomorphism relation is not Borel complete

**More importantly:** potential cardinality gives a way to show the nonexistence of a Borel reduction, even when the underlying isomorphism relation is not Borel.

**Side benefit:** the proof was model-theoretic, rather than forcing-theoretic.

**Note:** after naming  $\text{acl}(\emptyset)$ , the theory is Borel (in fact  $\Pi_3^0$ ).



# Koerwien's Example

The theory  $\mathbf{K}$  is in the language  $L = \{U, C_n, V_n, S_n, \pi_n : n \in \omega\}$ .  $\mathbf{K}$  states:

- $U$  and each of the  $V_n$  are infinite sorts;  $C_n$  is a sort of size two
- $\pi_n : V_n : U \times C_0 \times \cdots \times C_n$  is a surjection
- $S_n : V_n \rightarrow V_n$  is a successor function
- $\pi_n \circ S_n = \pi_n$

$\mathbf{K}$  is  $\aleph_0$ -stable, classifiable, and has depth two

$\mathbf{K}$  is not Borel, but  $\|\mathbf{K}\| = I_{\infty\omega}(\mathbf{K}) = \beth_2$ ;  $\mathbf{K}$  may not be grounded;  
It is unknown if  $\mathbf{K}$  and  $\mathbf{REF}$  are  $\leq_{\mathcal{B}}$ -comparable

**Note:**  $\text{Aut}(\text{acl}(\emptyset))$  is  $(2^\omega, +)$ , which is abelian;  
after naming  $\text{acl}(\emptyset)$ , isomorphism is  $\Pi_3^0$

# The Koerwien Tweak

The theory **TK** is in the language  $L = \{U, C_n, V_n, S_n, \pi_n, p_n : n \in \omega\}$ . TK states:

- $U$  and each of the  $V_n$  are infinite sorts;  $C_n$  is a sort of size  $2^n$
- $\pi_n : V_n \rightarrow U \times C_n$  is a surjection
- $p_n : C_{n+1} \rightarrow C_n$  is a two-to-one surjection
- $S_n : V_n \rightarrow V_n$  is a successor function
- $\pi_n \circ S_n = \pi_n$

TK is  $\aleph_0$ -stable, classifiable, and has depth two

TK is **Borel complete**, but  $I_{\infty\omega}(\text{TK}) = \beth_2$ , so not Borel and not grounded

**Note:** the only difference between TK and K is  $\text{Aut}(\text{acl}(\emptyset))$ ;  
Here  $\text{Aut}(\text{acl}(\emptyset))$  is  $\text{Aut}(2^{<\omega}, <)$ , which is highly nonabelian  
After naming  $\text{acl}(\emptyset)$ , K and TK become equivalent (so  $\Pi_3^0$ )

# Thank you!

The paper in question:  
[arXiv:1510.05679](https://arxiv.org/abs/1510.05679)