Countable Model Theory

and the Complexity of Isomorphism

Richard Rast

University of Maryland

November 21, 2014

Driving Questions

Everywhere: T is a complete first-order theory in a countable language L.

Question

What does it mean for the countable models of T to be complicated?

Question

What causes the countable models of T to be complicated?

Outline

Traditional Classification Theory

Classification theory says you have to have all of the following nice properties or else your model theory is "chaotic:"

- Stable (NSOP and NIP)
- Superstable
- NDOP
- Shallow

But these criteria do not give much information about the countable models.

Unstable Theories

Let T_1 be the theory of dense linear orders.

- T_1 has the strict order property with $x < y$, so is unstable.
- T_1 is $\aleph_0$-categorical.

Let T_2 be the theory of the random graph.

- T_2 has the independence property with xEy, so is unstable.
- T_2 is $\aleph_0$-categorical.

So instability does not imply "complicated countable models."

Non-superstable Theories

Hrushovski used a generalized Fraïsse limit (now called a Hrushovski construction) to construct an $\aleph_0$-categorical, strictly stable pseudoplane.

So (although the appropriate examples are complicated) un-superstability does not imply "complicated countable models."

DOP Theories

Let T be the "standard checkerboard example."
There are three infinite sorts A, B, and C, and $\pi : C \to A \times B$ is an infinite-to-one surjection.

- T is superstable (in fact, ω-stable).
- T has DOP.
- T is $\aleph_0$-categorical.

So DOP does not imply "complicated countable models."

Deep Theories

T is the "standard tree example."
There is a unary function f, an infinite-to-one function with no fixed points or cycles.

- T is superstable (in fact, ω-stable).
- T is NDOP (in fact, classifiable).
- T is deep.
- T has only countably many models.

So "deep" does not imply "complicated countable models."

Classifiable Theories?

Let $T = \mathrm{Th}(\mathbb{Z}, +)$.

- T is classifiable and shallow.
- T has the maximum number of countable models.

So "classifiable and shallow" does not imply "few countable models."
But actually, the models of T are not that hard to understand...

Beyond the Spectrum Problem

Not all theories with the maximum number of same models are equally complex:

- T_1 says E is an ER with infinitely many infinite classes, S is a successor function preserving E.
- T_2 says $<$ is a dense linear order and c_q are constants ordered by $\mathbb{Q}$.
- T_3 says $<$ is a discrete linear order without endpoints.

We can distinguish these by the complexity of the invariants we would use to classify them.

Borel Reducibility

Definition

Given two equivalence relations E and F on standard Borel spaces X and Y, say $E \leq_B F$ if there is a Borel $f : X \to Y$ where for all $a, b \in X$, $E(a, b)$ iff $F(f(a), f(b))$.

Say $E \sim_B F$ if $E \leq_B F$ and $F \leq_B E$.

Spaces of Models

$\mathrm{Mod}(L)$ is the set of L-structures with universe ω. The basic open sets are $\{M | M \models \phi(\overline{n})\}$ where $\phi(\overline{x})$ is an L-formula and $\overline{n}$ is a tuple from ω.

$\mathrm{Mod}(T) = \{M \in \mathrm{Mod}(L) : M \models T\}$ is a closed subspace of $\mathrm{Mod}(L)$, so is a standard Borel space.

$\cong_T$ is an invariant subset of $\mathrm{Mod}(T) \times \mathrm{Mod}(T)$. Compare complexity of theories by comparing $\leq_B$-complexity of their isomorphism relations.

Example

$(\mathrm{Mod}(T_1), \cong) <_B (\mathrm{Mod}(T_2), \cong) <_B (\mathrm{Mod}(T_3), \cong)$

For this talk: say $T_1 <_B T_2 <_B T_3$ for this.

Borel Completeness

E is *Borel complete* if *every* invariant F has $F \leq_B E$.

Theorem (Friedman, Stanley)

There are a lot of Borel complete classes: linear orders, graphs, trees, groups, fields...

Once you have a few examples, it's easy to get more by transitivity of $\leq_B$:

Example

- Bipartite graphs are Borel complete.
- $\mathrm{Th}(\mathbb{Z}, <)$ is Borel complete.

Π^0_α-completeness

E is Π^0_α-complete if, for every equivalence relation $F \in \Pi^0_\alpha$, $F \leq_B E$.

Theorem (Hjorth, Kechris, Louveau)

For every $\alpha < \omega_1$, there is a $\beta < \omega_1$ and an invariant equivalence relation $\cong_\beta$ which is Π^0_α and Π^0_α-complete.

Example

Let $T = \mathrm{Th}(\mathbb{Z}, S)$. Then $T \sim_B \cong_0 \sim_B (\mathbb{N}, =)$.

Example

Let $T = \mathrm{Th}(E, S)$ like before. Then $T \sim_B \cong_1 \sim_B (\mathbb{R}, =)$.

Example

Let $T = \mathrm{Th}(\mathbb{Q}, <, c_q)_{q \in \mathbb{Q}}$. Then $T \sim_B \cong_2$, so T is Π^0_3 and Π^0_3-complete.

A Working Definition of Complexity

We have two goals:

- to classify the "$\leq_B$ spectrum" for (countable complete first-order) theories, and
- to characterize when each possibility occurs.

Being $\aleph_0$-categorical is our "categoricity notion."
Being Borel complete is our "complete non-structure notion."
Being Π^0_α complete is an approximation to "non-structure."

General Results

Few results about this are known for general theories. Two exceptions:

Theorem (Ryll-Nardzewski)

$T \sim_B 1$ if and only if $S_n(T)$ is finite for every n.

Theorem (Marker)

If T is not small ($S(T)$ is uncountable) then $\cong_2 \leq_B T$.

Non-General Results

There are two major classes of theories where the $\sim_B$ structure has been heavily investigated and lines have been drawn:

- O-minimal theories
  - Existence of a nonsimple type
  - Size of $|S_1(T)|$
- $\aleph_0$-stable theories
  - ENI-DOP
  - ENI-deep

O-Minimal Theories

The $\sim_B$ problem for o-minimal T has been completely solved:

Theorem (R., Sahota)

If T is o-minimal, then exactly one of the following occurs:

- *T has exactly $3^a 6^b$ countable models, where $a, b < \aleph_0$.*
- *T is Borel equivalent to $\cong_1$.*
- *T is Borel equivalent to $\cong_2$.*
- *T is Borel complete.*

Each case is possible and characterized by syntactic properties of T.

Nonsimple Types

Definition (Mayer)

A nonalgebraic $p \in S_1(A)$ is *nonsimple* if there is an A-definable partial function $f : p^n \to p$ where for some $\overline{x} \in p^n$, $f(\overline{x}) \notin \overline{x}$.

Pertinent examples:

- Let $T = \mathrm{Th}(\mathbb{Q}, +, 0, 1, <)$. Let $p(x) = \{x > n : n \in \mathbb{Z}\}$.
  p is nonsimple under $x \mapsto 2x$.
- Let $T = \mathrm{Th}(\mathbb{R}, +, \cdot, <)$. Let $p = \mathrm{tp}(\pi)$.
  p is nonsimple under $(x, y) \mapsto (x + y)/2$.
- Let $T = \mathrm{Th}(\mathbb{Q}, f, <)$ where $f(x, y, z) = x + y - z$. Let $p = \{x = x\}$.
  p is nonsimple under $(x, y) \mapsto 2x - y$.

The Dividing Line

Let T be o-minimal in a countable language.

Definition

T *admits a nonsimple type* if there is a nonsimple $p \in S_1(A)$ for some A.

Equivalently: there is a nonsimple $p \in S_1(\emptyset)$.

Theorem

T is Borel complete if and only if T admits a nonsimple type.

Mayer's Theorem

Let T be o-minimal with no nonsimple types.

Theorem (Mayer)

If $M, N \models T$ are countable, then $M \cong N$ if and only if for all $p \in S_1(T)$, $(p(M), <) \cong (p(N), <)$.

A complete nonisolated $p \in S_1(A)$ is a *non-cut* if it has a supremum or infimum in $\mathrm{dcl}(A)$; otherwise it is a *cut*. There are three permissible order types for non-cuts, six for cuts, and one for isolated types. Therefore:

Corollary (Mayer)

Let a, b be the number of independent non-cuts and cuts (resp.) over $\emptyset$. If $a, b < \aleph_0$ then $I(T, \aleph_0) = 3^a 6^b$. Otherwise $I(T, \aleph_0) = 2^{\aleph_0}$.

No Nonsimple Types - The Answer

This can be made into a Borel map:

Theorem

Suppose T is o-minimal with no nonsimple types. Then:

- *If $a, b < \aleph_0$ then $T \sim_B (3^a 6^b, =)$*
- *If $S_1(T)$ and $a + b$ are countably infinite, then $T \sim_B \cong_1 \sim_B (\mathbb{R}, =)$.*
- *If $S_1(T)$ is uncountable then $T \sim_B \cong_2$.*

Since the cases are exhaustive, this completely answers this side of the question.

Archimedean Components

From now on, T is o-minimal and admits a nonsimple type.

Definition

Let $p \in S_1(A)$ be nonalgebraic. Say $a, b \models p$ are *Archimedean equivalent over* A if b is bounded by elements of $\mathrm{dcl}^p(Aa)$.

Denote this by $a \sim b$, or $a \sim_A b$ if A is unclear from context.

Definition

For any $p \in S_1(A)$ and model $M \supset A$, $(p(M)/\sim, <)$ is the *p-ladder* of M.

Faithful Types

Definition

A nonsimple $p \in S_1(A)$ is *faithful* if, whenever $\overline{a}$ realizes p^n and $[a_1] < \cdots < [a_n]$, if $b \in \mathrm{dcl}^p(A\overline{a})$, then $b \sim a_i$ for some $i \leq n$.

Some examples:

- $(\omega, <)$ and $p(x) = \{x > n : n \in \omega\}$
- $(\mathbb{Q}, +, <)$ and $p(x) = \{x > 0\}$

Some non-examples:

- $(\mathbb{R}, +, \cdot, <)$ and $p(x) = \mathrm{tp}(\pi)$
- $(\mathbb{Q}, f, <)$ where $f(x, y, z) = x + y - z$, and $p(x) = \{x = x\}$.

Faithfulness Implies Borel Completeness

Definition

A nonsimple $p \in S_1(A)$ is *faithful* if, for all $\overline{a} \in p^n$ and all $b \in \mathrm{dcl}^p(\overline{a})$, $b \sim a$ for some $a \in \overline{a}$.

Theorem

If T admits a nonsimple faithful $p \in S_1(T)$, then T is Borel complete.

Proof.

Let $(L, <)$ be a countable linear order. Let $X_L = \{c_\alpha : \alpha \in L\}$ be realizations of p with $[c_\alpha] < [c_\beta]$ for $\alpha < \beta$ in L. Let $\mathcal{M}_L = \Pr(X_L)$. Then $(p(\mathcal{M}_L)/\sim, <)$ is isomorphic to L, so $(\mathrm{LO}, <) \leq_B T$. □

The hard part is coming up with one.

Nonisolated Types

Lemma

Nonsimple non-cuts are faithful.

Lemma

If $p \in S_1(A)$ is a nonsimple cut, then either p is faithful or there is a nonsimple non-cut $q \in S_1(A)$.

So if there is a nonsimple nonisolated type in $S_1(T)$, T is Borel complete.

Unfaithful Types

Example

Let $T = \mathrm{Th}(\mathbb{Q}, <, f)$, where $f(x, y, z) = x + y - z$. Then $x = x$ is an isolated, nonsimple type which is not faithful.

Since $x = x$ is atomic, there are no other types to choose from, either.

If we pick two parameters (call them 0 and 1) then we get a nonsimple (faithful) non-cut at "infinity" where we can build a ladder. But it will *not* be preserved under general isomorphism.

Tails

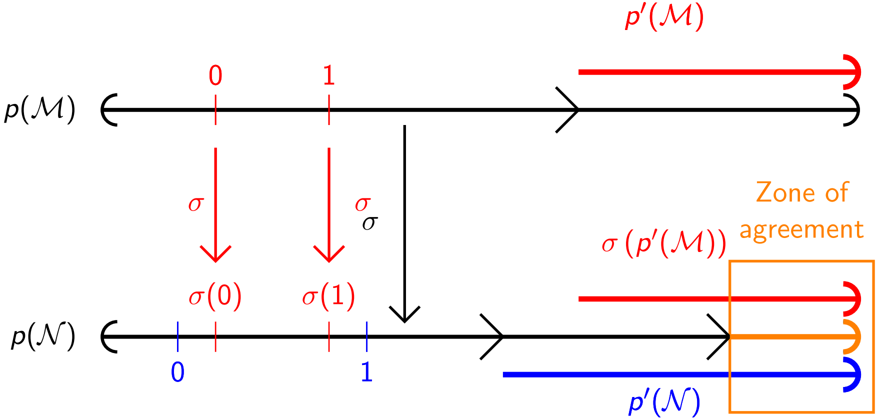
Example

Let $T = \mathrm{Th}(\mathbb{Q}, <, f)$, where $f(x, y, z) = x + y - z$. Then $x = x$ is an isolated, nonsimple type which is not faithful.

Given two parameter choices $\{0, 1\}$ and $\{0', 1'\}$, if x and y are "big enough" – infinite with respect to the quadruple $\{0, 0', 1, 1'\}$ – then $x \sim y$ over $(0, 1)$ if and only if $x \sim y$ over $(0', 1')$.
So if we fix $\{0, 1\}$, then build a long enough ladder above them, a *tail* of our intended linear order is preserved under isomorphism.

The Tail Picture

So suppose $p(x)$ is $x = x$, $p'(x)$ is the infinite extension of p to a $\{0,1\}$-type, and $\sigma : \mathcal{M} \to \mathcal{N}$ is an isomorphism. We get this:



So there is a common *tail* of the two ladders.

The General Proof

Lemma

Suppose $p \in S_1(T)$ is isolated and n-nonsimple and $\overline{a}, \overline{b}$ are from p^n. For all x, y realizing p and "infinite in $\overline{ab}$," $x \sim_{\overline{a}} y$ iff $x \sim_{\overline{b}} y$.

Fix a nonsimple isolated type $p \in S_1(T)$. We would *like* to build a Borel reduction from LO to $(\mathrm{Mod}(T), \cong)$ as follows:

1. Fix a set $A = \{1, \ldots, n\}$ of parameters to get a non-cut.
2. For a countable linear order L, fix a set $X_L = \{x_\alpha : \alpha \in L\}$ from p, where $x_\alpha > \mathrm{cl}^p_A(X_\alpha)$ for each α.
3. Let $\mathcal{M}_L \models T$ be constructible over $A \cup X_L$.
4. The map $\alpha \mapsto [x_\alpha]$ *will be* an order-isomorphism $L \to p'(\mathcal{M}_L)/\sim_A$.
5. For any orders L and L', if $\mathcal{M}_L \cong \mathcal{M}_{L'}$, then L and L' are isomorphic *on a tail*.

The Last Piece

Lemma

There is a Borel reduction $f : LO \to LO$ such that on the image of f, $\cong$ is equivalent to tail isomorphism.

Proof.

Let $A = \{0\} \cup [1,2]_{\mathbb{Q}} \cup \{3\}$. Then use $L \mapsto \omega \times [(L \times A) \cup \{\infty\}]$ □

Theorem

Let T be countable and o-minimal. Then T is Borel complete if and only if T admits a nonsimple type.

So the $\sim_B$ class of T can be completely determined by type counting and whether or not there is a nonsimple type.

$\aleph_0$-Stable Theories

Theorem (Laskowski, Shelah)

If T is $\aleph_0$-stable and ENI-DOP or ENI-deep, then T is Borel complete.

Example

DCF_0 is $\aleph_0$-stable with ENI-DOP, so is Borel complete.

A strongly regular p is *ENI* (eventually nonisolated) if it has finite dimension in some model.

The conditions “ENI-DOP” and “ENI-deep” are essentially the same as “DOP” and “deep,” but with some sort of ENI witness tied into it.

ENI-DOP

ENI-DOP is, essentially, a witness to DOP (the dimensional order property) with an ENI witnessing type.

Theorem (Laskowski, Shelah)

If T is $\aleph_0$-stable with ENI-DOP, then T is Borel complete.

Example

The "standard checkerboard example" with an ENI patch.

Moreover, the negation ENI-NDOP implies a *very weak* uniqueness theorem for ENI-active decompositions of models.

ENI-deep

Definition

An $\aleph_0$-theory T is *ENI-deep* if it admits a non-well-founded ENI-active decomposition tree.

Otherwise define the ENI-depth of T as the highest rank of any such tree.

Theorem (Laskowski, Shelah)

If T is $\aleph_0$-stable, ENI-NDOP, and ENI-deep, then T is Borel-complete.

Example

The "standard tree example" with an ENI patch.

Further Results

Corollary (R.)

If T is $\aleph_0$-stable, ENI-NDOP, and has ENI-depth $\geq 2 + \alpha$, then $\cong_\alpha \leq_B T$. This lower bound is sharp for all countable α.

If the ENI-depth is 0, then T is $\aleph_0$-categorical.
If the ENI-depth is 1, then T is either $\cong_0$ or $\cong_1$.

An example due to Koerwien is $\aleph_0$-stable, ENI-NDOP with ENI-depth 2, but is not Borel. It is unknown whether it is Borel complete.

The $\sim_B$ problem for $\aleph_0$-stable T remains otherwise open.

The Unknown

How about some open questions?

Parameters, I

The o-minimal theorem's proof originally looked like this:

Theorem (Sahota)

Suppose T admits a nonsimple type $p \in S_1(A)$ for some finite A. Then for some finite $B \supset A$, T admits a faithful nonsimple type over B, so T_B is Borel complete.

About a year later, the full theorem was proven with a lot more machinery:

Theorem (R.)

Suppose T admits a nonsimple type $p \in S_1(A)$ for some finite A. Then T is Borel complete.

Parameter removal is hard.

Parameters, II

The same thing happened in the $\aleph_0$-stable Borel completeness theorem. It is "much easier" to prove Borel completeness (in both the ENI-DOP and ENI-deep cases) after adding finitely many constants.

Question

Suppose T is Borel complete after adding finitely many constants. Must T be Borel complete?

If Borel completeness is really a "model-theoretic property" we might expect a positive answer. There are no known counterexamples.

Parameters, III

Question

How much can the $\leq_B$ complexity *change* by adding finitely many constants?

In the o-minimal case: adding finitely many constants can *increase* the number of models (finite to finite), but at most 3^a6^b to 3^{a+2b}.

In the $\aleph_0$-stable case: having ENI-DOP and the ENI-depth are unchanged by adding constants, but the question itself is not answered.

There are no known examples of the complexity going *down* by adding finitely many constants.

There are no general results (known to me) except for model counting theorems.

Dividing Lines

The best, vaguest, biggest question of the day:

Question

Are there classification-theory-like dividing lines for countable complexity?

The Ryll-Nardzewski theorem and Marker's non-small theorem are general enough, but neither gives a "bad implies maximally bad" condition.

The "admits a nonsimple type" condition is a good dividing line for countable complexity, but is only useful for o-minimal theories. Likewise for ENI-DOP and ENI-depth with $\aleph_0$-stable theories.

Otherwise, we don't know...