Potential Cardinality

or: Pretend it works and see where it gets you

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An Excellent Question

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   1. The class of countable graphs
   2. The class of countable $\mathbb{Q}$-vector spaces

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Borel Reductions

Let (X, E) and (Y, F) be equivalence relations on standard Borel spaces.

Definition

Say $(X, E) \leq_B (Y, F)$ if there is a function $f : X \to Y$ satisfying:

- f is Borel
- For all $a, b \in X$, aEb iff $faFfb$

Think: (X, E) is at most as complicated as (Y, F)

First Examples

If $\Phi \in L_{\omega_1\omega}$, then $\mathrm{Mod}(\Phi)$ is a Polish (standard Borel) space.

- Let X be the space of countable $\mathbb{Q}$-vector spaces.
- Let Y be the space of countable sets of countable $\mathbb{Q}$-vector spaces.
- Let Z be the space of countable graphs.

$$(X, \cong) <_B (Y, \cong) <_B (Z, \cong)$$

The Low End: Borel Relations

Fact: If $(X, E) \leq_B (Y, F)$ and F is a Borel subset of $Y \times Y$, then E is also Borel.

Some examples: All the following are Borel and equivalent to $(\mathrm{Mod}(T), \cong)$ for some appropriate first-order T:

0. $\cong_0$: Integers, up to equality
1. $\cong_1$: Real numbers, up to equality
2. $\cong_2$: Countable sets of reals, up to equality
3. $\cong_3$: Countable sets of countable sets of reals, up to equality

$\vdots$

Fact: $\cong_\alpha <_B \cong_\beta$ whenever $\alpha < \beta$.

Fact: $\cong_\phi$ is Borel if and only if $\mathrm{sr}(\phi) < \omega_1$
if and only if $\cong_\phi \leq_B \cong_\alpha$ for some $\alpha < \omega_1$

The Upper Edge: Borel Completeness

Definition

Say ϕ is Borel complete if it is $\leq_B$-maximal.
That is, for all ψ, $\psi \leq_B \phi$.

Theorem (Friedman, Stanley)

Lots of things are Borel complete. Things like linear orders, graphs, fields, groups, trees, ...

Evidently if ϕ is Borel complete, $\cong_\phi$ is not Borel.

Excellent question: Suppose $\phi \in L_{\omega_1 \omega}$. Must $\cong_\phi$ be either Borel or Borel complete? What if ϕ is first-order?

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2. If $\cong_\psi$ and $\cong_\phi$ are both Borel, there are some fairly coarse tools.

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These never seem to apply to first-order examples, for some reason.

A Pertinent Example

Let (X, E) be as usual, and let (X^ω, E^ω) be the jump:

$$\text{Let } \overline{x} = \{x_n : n \in \omega\} \text{ and } \overline{y} = \{y_n : n \in \omega\} \text{ so } \overline{x}, \overline{y} \in X^\omega.$$
$$\overline{x} E^\omega \overline{y} \text{ iff there is a } \sigma \in S_\infty \text{ where } x_n = y_{\sigma(n)} \text{ for all } n.$$

Theorem (Friedman, Stanley)

If (X, E) is as usual, $E \subset X \times X$ is Borel, and E has more than one class, then $(X, E) <_B (X^\omega, E^\omega)$.

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- No such G exists, since E^ω is Borel

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- $2^\kappa - 1 > \kappa$, so you can't reduce X^ω/E^ω to X/E.

Roadmap

Potentiality

Let A be any set. Let $\mathbb{V}[G]$ collapse $|\mathrm{trcl}(A)|$.
Then A is hereditarily countable in $\mathbb{V}[G]$, as well as in any $\mathbb{V}[G][H]$.
Phrased another way:

Every set is potentially hereditarily countable.

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Phrased another way:

Every set is potentially hereditarily countable.

Let α be any ordinal; then α is "potentially in ω_1."
But if A is not an ordinal, A is still not an ordinal in $\mathbb{V}[G]$, so A is not potentially in ω_1.

Sets are potentially in ω_1 iff they are ordinals.

Making it Rigorous

Let $\phi(x)$ be a (meta)-formula with parameters from HC. Say ϕ is a strong definition if its truth (persistently) does not change under forcing.

Precisely:

$$\text{For any } \mathbb{V}[G], \text{ any } a \in \mathrm{HC}^{\mathbb{V}[G]} \text{ and any } \mathbb{V}[G][H],$$
$$\mathrm{HC}^{\mathbb{V}[G]} \models \phi(a) \text{ iff } \mathrm{HC}^{\mathbb{V}[G][H]} \models \phi(a).$$

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Let a be *any* set. Say a potentially satisfies ϕ if, for some (any) forcing extension $\mathbb{V}[G]$ in which a is hereditarily countable, $\mathrm{HC}^{\mathbb{V}[G]} \models \phi(a)$.

The potential class ϕ_{ptl} is the set of all a which potentially satisfy ϕ.

It's Easier than It Sounds

Some examples:

- $\mathrm{HC}_{\mathrm{ptl}}$ is $\mathbb{V}$
- $(\omega_1)_{\mathrm{ptl}}$ is ON
- ω_{ptl} is ω
- $\mathbb{R}_{\mathrm{ptl}}$ is $\mathbb{R}$

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Some more:

- If X is strongly definable, the potential class of "countable sets of elements of X" is $\mathcal{P}(X_{\mathrm{ptl}})$
- If X and Y are strongly definable, $(X^Y)_{\mathrm{ptl}}$ is $(X_{\mathrm{ptl}})^{Y_{\mathrm{ptl}}}$
- If $\{X_i : i \in I\}$ are strongly definable, $(\bigcup_{i\in I} X_i) = \bigcup_{i\in I_{\mathrm{ptl}}}(X_i)_{\mathrm{ptl}}$

Potential Cardinality

Proposition

If $f : X \to Y$ is an injection (persistently, and everything is strongly definable) then $f_{\mathrm{ptl}} : X_{\mathrm{ptl}} \to Y_{\mathrm{ptl}}$ is also an injection.

If X is strongly definable, define the potential cardinality of X as $|X_{\mathrm{ptl}}|$.

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Some examples:

- $\|\mathbb{R}\| = \beth_1$
- $\|\mathcal{P}_{\aleph_1}(\mathbb{R})\| = \beth_2$
- $\|\omega_1\| = \infty$
- $\|\mathcal{P}_{\aleph_1}(X)\| = 2^{\|X\|}$
- $\|X^Y\| = \|X\|^{\|Y\|}$
- $\|\bigcup_{i \in I} X_i\| = \|I\| + \sup_{i \in I} \|X_i\|$

Roadmap

Scott Sentences, More Generally

We can define canonical Scott sentences for any model M in the usual way. Call this sentence $\mathrm{css}(M)$; note $\mathrm{css}(M) \in L_{|M|^+\omega}$.

Theorem

Let M and N be L-structures. The following are equivalent:

1. $\mathrm{css}(M) = \mathrm{css}(N)$
2. $N \models \mathrm{css}(M)$
3. M and N are back-and-forth equivalent.
4. M and N are potentially isomorphic.

Scott Sentences, Most Generally

A canonical Scott sentence extending ϕ is an $L_{\infty\omega}$-sentence ψ satisfying all the following:

- ψ fits the syntactic form of a canonical Scott sentence.
- ψ is not formally inconsistent.
- $\psi \wedge \neg\phi$ is formally inconsistent.

Fact: these conditions are equivalent to "in some (any) forcing extension in which $\phi \wedge \psi \in L_{\omega_1\omega}$, ψ is the Scott sentence of a countable model of ϕ."

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Fact: $\mathrm{CSS}(\phi)_{\mathrm{ptl}}$ is the set of all canonical Scott sentences extending ϕ.

Warning: canonical Scott sentences may not have models in $\mathbb{V}$.

The Actual Point of All This Machinery

Theorem

If $f : \mathrm{Mod}(\Phi_1) \leq_B \mathrm{Mod}(\Phi_2)$, then the map $\mathrm{css}(M) \mapsto \mathrm{css}(f(M))$ is a persistent strongly definable injection.

So define $\|\Phi\|$ as $|\mathrm{CSS}(\Phi)_{\mathrm{ptl}}|$.

Corollary

If $\|\Phi\| > \|\Psi\|$, then $\mathrm{Mod}(\Phi) \not\leq_B \mathrm{Mod}(\Psi)$.

A Simple Consequence

For any ϕ, let $I_{\infty\omega}(\phi)$ be the number of back-and-forth inequivalent models of ϕ.

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If isomorphism for ϕ is Borel, then $I_{\infty\omega}(\phi) < \beth_{\omega_1}$.

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- By an easy induction on α, $\| \cong_\alpha \| = \beth_{-1+\alpha+1}$
- $I_{\infty\omega}(\phi) \leq \|\phi\| \leq \| \cong_\alpha \| = \beth_{-1+\alpha+1} < \beth_{\omega_1}$

Roadmap

Axioms for an Example

Let $L = \{E_n : n \in \omega\}$. REF will be the L-theory with the following axioms:

- Each E_n is an equivalence class with 2^n classes.
- Each E_{n+1} refines E_n.
- Each E_n-class splits into exactly two E_{n+1}-classes.

Proposition

REF is complete with quantifier elimination and a prime model. It is small, superstable, and not ω-stable.

REF Is Not Borel

Fact: $\cong_\phi$ is Borel if and only if, for some $\alpha < \omega_1$, $\equiv_\alpha$ implies isomorphism for countable models of ϕ.

Proposition

Isomorphism for REF is not Borel.

Proof outline:

- Since REF is complete with more than one model, $\equiv_0$ does not imply isomorphism.
- Suppose $A, B \models \mathrm{REF}$ are countable, $A \equiv_\alpha B$, and $A \ncong B$.
- Let X and Y be disjoint countable dense subsets of 2^ω.
- Construct M_X and M_Y countable where $M_X \equiv_{\alpha+1} M_Y$ but $M_X \ncong M_Y$.
- Similar construction at limit stages.

Coding a Bit of Complexity

Prop: $\cong_2 \leq_B \mathrm{REF}$
Proof outline:

1. Pick a prime model of REF; label its elements by $2^{<\omega}$
2. Fix an enumeration $f : 2^{<\omega} \to \omega$; expand each element η to have color $f(\eta) + 1$
3. Given $X \subset 2^\omega$ countable, for each $\eta \in X$, add new elements a_η with E_∞ class η and color ∞
4. Call the result M_X
5. If $M_X \cong M_Y$, then the isomorphism preserves colors, so $X = Y$ (and conversely)

Corollary: $I_{\infty\omega}(\mathrm{REF}) \geq \beth_2$.
Proof: Leave off the word "countable" in step 3.

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Proof:

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Warning: $I_{\infty\omega}(\phi) \leq_B \|\phi\|$ but this is strict in general.
So this gives us no information about Borel reducibility on its own.

Consistent Implies Satisfiable

Difficult Fact: If $\phi \in \mathrm{CSS}(\mathrm{REF})_{\mathrm{ptl}}$, then ϕ has a model.
Proof Idea:

- Give a concise list of invariants of a model, called data.

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- Show that $M \equiv_{\infty\omega} N$, where N is in $\mathbb{V}[G]$ and $N \models \phi$.
- Conclude that $M \models \phi$.

Thus $\|\mathrm{REF}\| = I_{\infty\omega}(\mathrm{REF})$.

REF Is Not Borel Complete

Theorem

$\cong_3 \not\leq_B \mathrm{REF}$

Proof: $\| \cong_3 \| = \beth_3$, while $\|\mathrm{REF}\| = I_{\infty\omega}(\mathrm{REF}) = \beth_2 < \beth_3$.

Corollary

There is a first-order theory whose isomorphism relation is neither Borel nor Borel complete.

Extensions

Corollary: For every ordinal $2 \leq \alpha < \omega_1$, there is a complete first-order theory T_α where:

- $\cong_\alpha \leq_B T_\alpha$
- Isomorphism for T_α is not Borel
- $\cong_{\alpha+1} \not\leq_B T_\alpha$, and in particular T_α is not Borel complete.

Open: Is the above possible for $\alpha = 0$ or $\alpha = 1$?

The case $\alpha = 1$ is known to be possible for $L_{\omega_1 \omega}$-sentences (eg: abelian p-groups), but is still open for first-order theories.

The case $\alpha = 0$ is exactly Vaught's conjecture.