Linear Orderings
and the Complexity of Isomorphism

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April 14, 2015

The Old Theorem

Theorem (Rubin)

Let $\mathcal{A}$ be a linear ordering, possibly with countably many unary predicates attached. Let $T = Th(\mathcal{A})$. Then T satisfies Vaught's conjecture. In particular, T has only finitely many or continuum-many models.

If there are only finitely many unary predicates, then T is either $\aleph_0$-categorical or has continuum-many models.

Our theorem is heavily based on the proof of this theorem.

The New Version

Theorem (R.)

Let $\mathcal{A}$ be a linear ordering, possibly with countable many unary predicates attached. Let $T = Th(\mathcal{A})$. Then T is smooth, Borel equivalent to $\cong_1$, or Borel complete.

If there are only finitely many unary predicates, then T is either $\aleph_0$-categorical or Borel complete.

Important – having many models is *not* the interesting dividing line!

Overview of the Proof

If $\mathcal{A}$ is *self-additive*, then $\mathrm{Th}(A)$ is either $\aleph_0$-categorical or Borel complete.

General $\mathcal{A}$ can be "definably divided" into convex self-additive pieces; call the partition $(I, <)$.

If one of the pieces is Borel complete, so is $\mathrm{Th}(\mathcal{A})$. Otherwise:

- If $\mathrm{CB}(I) = 0$, $\mathrm{Th}(\mathcal{A})$ is $\aleph_0$-categorical.
- If $\mathrm{CB}(I) = 1$, $\mathrm{Th}(\mathcal{A})$ has $n \geq 3$ countable models.
- If $2 \leq \mathrm{CB}(I) < \infty$, $\mathrm{Th}(\mathcal{A})$ is $\cong_1$ (equality on reals).
- If $\mathrm{CB}(I) = \infty$, $\mathrm{Th}(\mathcal{A})$ is $\cong_2$ (countable sets of reals).

A quick refresher on the Borel complexity stuff…

Spaces of Models

$\mathrm{Mod}(L)$ is the set of L-structures with universe ω. The basic open sets are $\{M | M \models \phi(\overline{n})\}$ where $\phi(\overline{x})$ is an L-formula and $\overline{n}$ is a tuple from ω.

$\mathrm{Mod}(T) = \{M \in \mathrm{Mod}(L) : M \models T\}$ is a closed subspace of $\mathrm{Mod}(L)$, so is a standard Borel space.

There is a natural action of (a closed subgroup of) S_∞ on $\mathrm{Mod}(L)$ which preserves $\mathrm{Mod}(T)$, so $\mathrm{Mod}(T)$ is *invariant*.

Theorem (Lopez-Escobar)

If $X \subset Mod(L)$ is invariant and Borel, then X is $Mod(\phi)$ for some sentence $\phi \in L_{\omega_1,\omega}$ (possibly expanding the language, depending on your definition of invariant)

Complexity of Theories

Definition

A *Borel reduction* from (X, E) to (Y, F) is a Borel function $f : X \to Y$ where for all $x, x' \in X$, $(x, x') \in E$ if and only if $(fx, fx') \in F$.

If there is such an f, say $(X, E) \leq_B (Y, F)$.

We say $T_1 \leq_B T_2$ when we really mean
$(\mathrm{Mod}(T_1), \cong_{T_1}) \leq_B (\mathrm{Mod}(T_2), \cong_{T_2})$.

$\leq_B$ is our way of comparing complexity.

Borel Completeness

(X, E) is *Borel complete* if *every* invariant Borel (Y, F) has $F \leq_B E$.

Theorem (Friedman, Stanley)

There are a lot of Borel complete classes: linear orders, graphs, trees, groups, fields...

Once you have a few examples, it's easy to get more by transitivity of $\leq_B$:

Example

- Bipartite graphs are Borel complete.
- $\mathrm{Th}(\mathbb{Z}, <)$ is Borel complete.

Some "Minimal" Examples

Theorem

If (X, E) is Borel complete, $E \subset X \times X$ is not Borel.
If $(X, E) \leq_B (Y, F)$ and F is Borel, then E is also Borel.

Borel relations are "fairly tame."

For every $n \in \omega$, $(n, =)$ is the $\sim_B$-unique Borel relation with n classes. It is minimal among the relations with $\geq n$ classes.

Equality on a standard Borel space has several names: $(2^{\mathbb{N}}, =)$, $(\mathbb{R}, =)$, $\cong_1$. It is minimal among the Borel relations with uncountably-many classes.

"Countable subsets of the reals" has several names: E^{set}, F_2, $\cong_2$.

Theorem (Marker)

$\cong_2$ is minimal among isomorphism relations for non-small theories.

A Working Definition of Complexity

We have two goals:

- to classify the "$\leq_B$ spectrum" for (countable complete first-order) theories, and
- to characterize when each possibility occurs.

Being $\aleph_0$-categorical is "simplest possible."

Being Borel complete is "most complex possible."

On to the theorem at hand...

$\aleph_0$-Categorical Orders

Definition

Say $\mathcal{A}$ is $\aleph_0$-categorical if for all countable $\mathcal{B}$, if $\mathcal{B} \equiv \mathcal{A}$, then $\mathcal{B} \cong \mathcal{A}$.

In particular, if $\mathcal{A}$ is finite, $\mathcal{A}$ is $\aleph_0$-categorical.

Let $\mathcal{S}_0$ be the set of 1-point orders.
Let $\mathcal{S}_{n+1}$ contain all the following:

- $\mathcal{S}_n$
- If $\mathcal{A}, \mathcal{B} \in \mathcal{S}_n$, then $\mathcal{A} + \mathcal{B} \in \mathcal{S}_{n+1}$
- If $\mathcal{A}_1, \ldots, \mathcal{A}_k \in \mathcal{S}_n$, then $\sigma(\mathcal{A}_1, \ldots, \mathcal{A}_k) \in \mathcal{S}_{n+1}$

Theorem (Rosenstein; Mwesigye, Truss)

$\mathcal{A}$ is $\aleph_0$-categorical if and only if $\mathcal{A} \cong \mathcal{B}$ for some $\mathcal{B} \in \mathcal{S} = \bigcup_n \mathcal{S}_n$.

Useful Facts

If $\mathcal{A}$ is $\aleph_0$-categorical, let $r(\mathcal{A})$ be the first n where $\mathcal{A} \in \mathcal{S}_n$.

Proposition

If $\mathcal{B}$ is a convex subset of $\mathcal{A}$, then $r(\mathcal{B}) \leq 2r(\mathcal{A}) + 1$.
Further, there are only finitely many convex subsets of $\mathcal{A}$, up to $\cong$.

Proposition

If L is finite:
Every $\mathcal{S}_n$ is finite and every $\mathcal{A} \in \mathcal{S}$ has $Th(\mathcal{A})$ finitely axiomatizable.

Consequently, there is a single sentence σ_n stating "I am not an $\aleph_0$-categorical order of rank at most n."

Self-Additive Orders

Fact (Rubin)

The following are equivalent for any $\mathcal{A}$ with more than one point:

- *The canonical embeddings $\mathcal{A} \to \mathcal{A} + \mathcal{A}$ are elementary,*
- *If $\mathcal{B} \equiv \mathcal{C} \equiv \mathcal{A}$, then the canonical embeddings $\mathcal{B}, \mathcal{C} \to \mathcal{B} + \mathcal{C}$ are elementary,*
- *Infinite versions of the preceding, and*
- *For every formula $\phi(x)$, if $\phi(\mathcal{A})$ is convex, then it's either $\mathcal{A}$ or $\emptyset$.*

If $\mathcal{A}$ satisfies any of the above, call it *self-additive*.

Some examples:

- $(\mathbb{Z}, <)$ is self-additive, while
- $(\mathbb{Z} + \mathbb{Q}, <)$ is not.

The Bounded Equivalence Relation

Let $a, b \in \mathcal{A}$ which is self-additive. Say $a \in \mathbf{c}_D(b)$ if there is a formula $\phi(x, y)$ where $\phi(x, b)$ is convex, bounded, and contains both a and b.

Fact

If $\mathcal{A}$ is self-additive, the relation $x \in \mathbf{c}_D(y)$ is an equivalence relation with convex classes. (in fact this is equivalent to self-additivity)

Examples:

- In $\mathcal{A} = (\mathbb{Q}, <)$, $\mathbf{c}_D(a) = \{a\}$, so $\mathcal{A}/\mathbf{c}_D \cong \mathbb{Q}$.
- In $\mathcal{B} = (L \times \mathbb{Z}, <)$, $\mathbf{c}_D(a) = \{S^n(a) : n \in \mathbb{Z}\}$, so $\mathcal{A}/\mathbf{c}_D \cong L$.

The Beginnings of Complexity

Lemma

If $\mathcal{A}$ is self-additive, $T = \mathrm{Th}(\mathcal{A})$, and $S_1(T)$ is infinite, then T is Borel complete.

Outline of proof:

- Let $p \in S_1(T)$ be nonisolated.
- Let $\mathcal{B} \models T$ omit p and let $\mathcal{C} \models T$ realize p at c.
- Let $\mathcal{C}'$ be the $\mathbf{c}_D$-class of c.
- The structure $I := \mathcal{B} + \mathcal{C}' + \mathcal{B}$ realizes T and has exactly one $\mathbf{c}_D$ class touching p.
- The map $L \mapsto L \times I$ is a Borel reduction from linear orders to T. Why? The set of all $\mathbf{c}_D$-classes containing a realization of p is an $\cong$-invariant and is order-isomorphic to L.

Finitely Many 1-Types

The next step is this:

Lemma

Say $T = Th(\mathcal{A})$. If $S_1(T)$ is finite, then T is $\aleph_0$-categorical or Borel complete.

This goes by induction on $n = |S_1(T)|$.

If $n = 1$, T is self-additive, so is either $(\mathbb{Q}, <)$ or $(\mathbb{Z}, <)$ (with some uniform color), done.

The $n + 1$ step goes by cases. First, say T is not self-additive. Then:

- Let $\phi(x)$ be a proper initial formula.
- $\phi(\mathcal{A})$ and $\neg\phi(\mathcal{A})$ are both either $\aleph_0$-categorical or Borel complete.
- If both are $\aleph_0$-categorical, so is T [Rosenstein]
- If either is Borel complete, so is T (direct sum construction)

Finitely Many 1-types (cont.)

Fact

Suppose $\mathcal{A}$ is self-additive, $T = Th(\mathcal{A})$, and $S_1(T)$ is finite. Then one of two things happens:

- *There is an $a \in \mathcal{A}$ where $\mathbf{c}_D(a) \prec \mathcal{A}$*
- *For all a, $\mathbf{c}_D(a)$ is a-definable, and $\mathcal{A}/\mathbf{c}_D \cong (\mathbb{Q}, <)$.*

  In fact, up to $\equiv$, $\mathcal{A}$ is a shuffle of some $D_1, \ldots, D_n$, and each of the D_i copies is a $\mathbf{c}_D$ component.

Finish the proof:

- If $\mathbf{c}_D(a) \prec I$, $L \mapsto L \times \mathbf{c}_D(a)$ shows T is Borel complete.
- Otherwise, if $n \geq 2$, inductive hypothesis applies to the D_i, and either...
  - All the D_i are $\aleph_0$-categorical, so T is [Rosenstein], or
  - One of the D_i is Borel complete, so T is (direct argument)
- Finally $n = 1$, so D_1 is not SA, and a previous case applies to D_1

What We Learned

If $\mathcal{A}$ is self-additive, $\mathrm{Th}(\mathcal{A})$ is either $\aleph_0$-categorical or Borel complete.

Interval Types

A *convex formula* is a formula $\phi(x)$ without parameters where $\phi(\mathcal{A})$ is a convex set.

A *convex type* is a complete consistent set of convex formulas.

The space $IT(T)$ is the set of all convex types.

Proposition

$IT(T)$ is a complete linear order and a compact Hausdorff space.
A type $\Phi \in IT(T)$ is isolated topologically if and only if it's isolated by a "complete convex formula".

Self-Additive Decompositions

Let $\mathfrak{C} \models T$ be $\aleph_0$-saturated (fix one for the rest of the talk).

Fact

For every $\Phi \in IT(T)$, $\Phi(\mathfrak{C})$ is self-additive or a single point.

Need saturation for technical reasons only.

Proposition

If some $T^{\mathfrak{C}}_{\Phi} := Th(\Phi(\mathfrak{C}))$ is Borel complete, so is T.

Proof:

- Pick a countable $\mathcal{A} \models T$ where $\mathcal{A} \prec \mathfrak{C}$ and $\Phi(\mathcal{A}) \prec \Phi(\mathfrak{C})$
- $T^{\mathfrak{C}}_{\Phi} = T^{\mathcal{A}}_{\Phi}$, so it's still Borel complete
- $T^{\mathcal{A}}_{\Phi} \leq_B T$ by sum decompositions

The Leftovers

For the rest of the proof, assume $T_\Phi^{\mathfrak{C}}$ is $\aleph_0$-categorical for every Φ.

Fact

For every Φ, the set $\{T_\Phi^{\mathcal{A}} : \mathcal{A} \models T\}$ is finite and every theory is $\aleph_0$-categorical (or is $Th(\emptyset)$).

Proof: It's *almost* true that $\Phi(\mathcal{A})$ is convexly embedded in $\Phi(\mathfrak{C})$.

Proposition

$T \leq_B \cong_2$ (countable sets of reals)

Proof outline:

- $\mathcal{A} \cong \mathcal{B}$ iff for every Φ, $\Phi(\mathcal{A}) \cong \Phi(\mathcal{B})$, iff for every Φ, $\Phi(\mathcal{A}) \equiv \Phi(\mathcal{B})$
- $IT(T)$ is a standard Borel space of size continuum
- Send $\mathcal{A} = \{a_1, a_2, a_3, \ldots\}$ to the sequence of pairs $\langle IT(a_n), T_{IT(a_n)}^{\mathcal{A}} \rangle$
- This is a Borel reduction $T \leq_B \cong_2$ by points 1 and 2

The Leftovers, Second Day

If $IT(T)$ is uncountable, T is not small, so $T \sim_B \cong_2$. So:
For the rest of the proof, assume $IT(T)$ is countable.

Proposition

$T \leq_B \cong_1$ (equality on the reals)

Proof outline:

- Fix an enumeration $\{\Phi_1, \Phi_2, \ldots\}$ of $IT(T)$
- $\mathcal{A} \cong \mathcal{B}$ if and only if, for every Φ, $\Phi(\mathcal{A}) \equiv \Phi(\mathcal{B})$
- Send $\mathcal{A}$ to the element of ω^ω given by $n \mapsto T_\Phi^\mathcal{A}$
- This is a Borel reduction $T \leq_B \cong_1$ (equality on ω^ω) by point 1

The Leftovers, Third Day

Fact

If $\{\Phi_i : i \in I\}$ is a discrete subset of $IT(T)$, then for any $\mathcal{A} \models T$ and any $X \subset I$, the set $\{a \in \mathcal{A} : \forall i \in X,\ A \models \neg\Phi_i(a)\}$ is an elementary substructure of $\mathcal{A}$.

The last bit of cleanup:

- $IT(T)$ is countable, so the CB rank is bounded.
- If $\mathrm{CB}(IT(T)) = 0$, every $\mathcal{A}$ is a finite sum of $\aleph_0$-categorical orders.
- If $\mathrm{CB}(IT(T)) = 1$, there are only finitely many finite choices.
- If $\mathrm{CB}(IT(T)) \geq 2$, there are infinitely many rank-one types, so
  - Pick $\mathcal{A} \prec \mathfrak{C}$ where $\Phi(A) \prec \Phi(\mathfrak{C})$ whenever $\mathrm{CB}(\Phi) = 1$
  - For every $X \subset \mathrm{CB}^{=1}(IT(T))$, there is $\mathcal{A}_X \models T$ as in the fact.
  - This is a Borel reduction $\cong_1 \leq_B T$

The End of the General Case

Theorem

Suppose $\mathcal{A}$ is a linear order with countably many unary predicates added. Let $T = Th(\mathcal{A})$. Then exactly one of the following happens:

- *T is Borel complete, because some self-additive piece is. Otherwise:*
- *$T \sim_B \cong_2$, because $IT(T)$ is uncountable.*
- *$T \sim_B \cong_1$, because $IT(T)$ is countable but has a rank-two type.*
- *T has finitely many models, because $IT(T)$ has only finitely many nonisolated types.*
- *T is $\aleph_0$-categorical, because it's a finite sum of self-additive pieces.*

Finite Languages

The promised special case:

Theorem

If L is finite, then T is $\aleph_0$-categorical or Borel complete.

Proof outline:

- Suppose T is not $\aleph_0$-categorical.
- Let $\Phi \in IT(T)$ be nonisolated, so $\Phi = \lim_n \Phi_n$ for some $\Phi_n \in IT(T)$.
- Let σ_k say "I am an $\aleph_0$-categorical linear order of rank at most k."
- There is a number $c(k)$ where σ_k implies there are only $c(k)$-many definable convex pieces.
- Any $\phi(x) \in \Phi(x)$ contains a pair $a < b$ where $[a, b]$ has infinitely many definable convex pieces.
- The type $\Phi(a) \wedge \Phi(b) \wedge \{[a, b] \models \neg\sigma_k : k \in \omega\}$ is consistent.
- $\Phi(\mathfrak{C})$ has a non-$\aleph_0$-categorical convex subset $[a, b]$.
- $T^{\mathfrak{C}}_{\Phi}$ is Borel complete.

What can we learn from this?

Under the Hood

What about all those "Facts?"

Lemma (M. Rubin)

Let $\mathcal{B} \subset \mathcal{A}$ be convex (not necessarily definable or even type-definable) For any $\phi(\overline{x})$ over $\mathcal{A}$, there is a $\phi^*(\overline{x})$ over $\mathcal{B}$ where for all $\overline{b}$ from $\mathcal{B}$,

$$\mathcal{A} \models \phi(\overline{b}) \textit{ if and only if } \mathcal{B} \models \phi^*(\overline{b})$$

Most of the "Facts" use this trick to show a model (constructed in a concrete way) is an elementary substructure of something else.

What's So Great About Linear Orders?

The main point is that every man is an island.
That is:

- Suppose $A = B + C$
- For every tuple $b, b' \in B$, where $\mathrm{tp}^B(b) = \mathrm{tp}^B(b')$, and every $c \in C$, $\mathrm{tp}^A(bc) = \mathrm{tp}^A(b'c)$
- No element from C can tell elements of B apart
- This applies to *any* convex decomposition of A

This is why we can toss in unary predicates.
This is also why we can alter models with so much freedom – if it agrees "locally" with the thing we started with, it agrees "globally."

In theory you can do this for any "convex decomposition" of any structure in any language, but there are not many interesting examples (even o-minimal structures).

Linear Orders and O-Minimal Theories are the Same

Our big theorems were:

Theorem

Suppose T is either an o-minimal theory or a colored linear order. Then T is either Borel complete or:

- *T is $\aleph_0$-categorical – if $S_1(T)$ is finite.*
- *T has finitely many countable models – if $S_1(T)$ has finitely many independent nonisolated types.*
- *$T \sim_B \cong_1$ – if $S_1(T)$ has countably many independent nonisolated types.*
- *$T \sim_B \cong_2$ – if $S_1(T)$ is uncountable.*

How deep does this parallel go?

Linear Orders Aren't O-Minimal

Theorem

If T is an o-minimal L-theory, then T is Borel complete if and only if for some finite $L_0 \subset L$, $T \restriction_{L_0}$ is Borel complete.

Is this true for colored linear orders?

No.

Example

Let $L = \{U_n : n \in \omega\}$, let T say that the U_n are all disjoint, dense, codense. T is Borel complete but every finite reduct is $\aleph_0$-categorical.

O-Minimal Theories Aren't Linear Orders

Theorem

If T is a colored linear order in a finite language, then T is $\aleph_0$-categorical or Borel complete.

Is this true for o-minimal theories?

No.

Example

Let $(I, <)$ be $\mathbb{Q}(\sqrt{2})$ with the inherited order. Add two unary functions:

- $f(x) = x + 1$, restricted to $x \in [0, 2]$
- $g(x) = x + \sqrt{2}$, restricted to $x \in [0, 2]$

What's So Great About Ordered Structures?

It's still strange that the main theorem is (more or less) the same for o-minimal theories as for colored linear orders.

Question

If $\mathcal{M} = (M, <, \ldots)$ is an ordered structure in a countable language, and $T = \mathrm{Th}(\mathcal{M})$. Must one of the following happen?

- $T <_B \cong_0$ (finitely many models)
- $T \sim_B \cong_1$ (exactly equality on $\mathbb{R}$)
- $T \sim_B \cong_2$ (countable subsets of $\mathbb{R}$)
- T is Borel complete

Probably not (this would be bizarre).
But the "usual tricks" don't work, and there's a frustrating lack of examples.

Obligatory Parameter Slide

In general it's really hard to answer the question: if you add a constant symbol to T, how does the complexity of the result compare to T?

Proposition

If you add a constant symbol to a typical structure $\mathcal{A}$, the number of finite models can go up (or stay the same), but no other changes can occur.

Essentially, (a) nothing interesting can happen, and (b) if you change, you go strictly *up* (a tiny bit).

It's known in general that you *can* go down (by settling a bunch of binary flips), but not too much. Specifically, $T \leq_B (T(a))^+$.
It's not known if you can go *up* in a significant way. There are no known upper bounds on $T(a)$ in terms of T.

The End

The most prominent citations:

- *Theories of Linear Order*, M. Rubin (M.Sc. Thesis)
- *Linear Orderings*, Rosenstein (Book)