The Borel Complexity of Isomorphism

for some First Order Theories

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Thesis Defense

March 29, 2016

Roadmap

Model Theory

MODEL THEORY is concerned with the following objective:

Given a theory T,

try to understand the models of T.

Sentences

For us, a sentence is a meaningful, finite expression using the following logical symbols:

$$\wedge, \vee, \rightarrow, \neg, \forall, \exists, (,)$$

Along with variables and symbols from a formal language.
Some examples:

- $L_{gp} = \{\cdot, ^{-1}, e\}$
- $L_{ring} = \{+, \cdot, -, 0, 1\}$
- $L_{ord} = \{<\}$
- $L_{orfld} = \{<, +, \cdot, -, 0, 1\}$

All languages are assumed to include $=$.

Sentences, II

Examples:

- $\forall x\ \forall y\ \ (x < y\ \ \to \exists z(x < z \wedge z < y))$
- $\forall c_0 \forall c_1 \cdots \forall c_n (\bigvee_{i=0}^{n} c_i \neq 0) \to \exists x(c_n x^n + \cdots + c_0 = 0)$

Caveats:

- (Compactness): Things like "there are only finitely many things where ..." are usually not expressible.
- Quantifiers range across elements of a specified set (the universe). We can't quantify across functions or subsets or etc.

With some cleverness we can sometimes get around these limitations.

Theories and Models

A theory is a collection of sentences in a specific language.

- For instance, let $\mathtt{RCF}$ be the theory of real-closed fields in the language $\{+, \cdot, 0, 1, <\}$.

Given a language L, an L-structure is a set with interpretations of the symbols of L.

- $(\mathbb{R}, +, \cdot, 0, 1, <)$ is an L-structure where $L = \{+, \cdot, 0, 1, <\}$

A model of a theory is an L-structure making all the sentences of the theory true.

- $(\mathbb{R}, +, \cdot, 0, 1, <)$ is a model of $\mathtt{RCF}$.

Countable Model Theory, I

Today we're talking about countable models of a theory. Why?

This is a natural class to work on:

- Easy to define and describe
- The uncountable models are already well-understood (Shelah, et. al.)

This is a useful class to work on:

- Existing results suggest a connection between the number of countable models and model-theoretic properties:
  - Ryll-Nardzewski: having a unique countable model is equivalent to "for all n, $S_n(T)$ is finite"
  - Marker: having some uncountable $S_n(T)$ implies the countable models are "fairly complicated"
- New results suggest dichotomies in some cases (e.g. ordered theories)

Roadmap

Understanding the Countable Models

For us, understanding the countable models means determining how difficult the isomorphism problem[1] is.

Examples:

- The problem for $\mathbb{Q}$-vector spaces is easy: just take a basis of each space, and see whether they're the same size.
- The problem for graphs (or groups, or fields...) is apparently hard.

This question is inherently comparative.

[1] Determining if two countable models are isomorphic.

The Complexity of Isomorphism

How do we measure the complexity of the isomorphism problem?

One classical idea was to count the number of countable models:

- $\mathbb{Q}$-vs has $\aleph_0$ countable models.
- RCF has $2^{\aleph_0}$ (continuum) countable models
- Groups has $2^{\aleph_0}$ countable models

This has lots of problems:

- There are only a few values that can possibly be the number: $\{1, \not{2}, 3, 4, 5, 6, 7 \ldots, \aleph_0, \aleph_1, 2^{\aleph_0}\}$
- Most interesting theories have $2^{\aleph_0}$ countable models

This fails to distinguish between things that should be distinguishable.

Borel Reductions

A better way is through Borel reductions.
Fix theories Φ and Ψ.

A Borel reduction from Φ to Ψ is a function which

1. takes countable models of Φ to models of Ψ, and
2. is injective on isomorphism classes, and
3. is "sufficiently mechanical."

Intuition: if Φ Borel reduces to Ψ, then the countable models of Φ are "less complicated" than the countable models of Ψ.

Condition (3) is needed to avoid trivialities.

Borel Reductions, Formally

Fix theories Φ and Ψ.

$\mathrm{Mod}_\omega(\Phi)$ and $\mathrm{Mod}_\omega(\Psi)$ are Polish spaces under the formula topology.

$f : \mathrm{Mod}_\omega(\Phi) \to \mathrm{Mod}_\omega(\Psi)$ is a Borel reduction if:

1. For all $M, N \models \Phi$, $M \cong N$ iff $f(M) \cong f(N)$
2. Preimages of Borel sets are Borel, in the formula topology.

Say $\Phi \leq_B \Psi$ if such an f exists.

Plainly: (2) means that if some property holds in $f(M)$, there is a logical reason for it in M.

A Real Example

Let Φ be "linear orders" and Ψ be "real closed fields." Then $\Phi \leq_B \Psi$.

Proof outline:

- Fix a linear order $(I, <)$.
- Pick a sequence $(a_i : i \in I)$ from a monster real closed field where $1 \ll a_i$ for all i, and if $i < j$, then $a_i \ll a_j$.
- Let M_I be the real closure of $\{a_i : i \in I\}$.
- $(I, <) \cong (J, <)$ iff $M_I \cong M_J$.
- f is "obviously Borel."

Establishing Some Benchmarks

Borel reducibility is inherently relative; it's hard to gauge complexity of (the countable models of) a sentence on its own.

We ameliorate this by establishing some benchmark sentences:

- which are distinguishable from each other, and
- whose countable models are easily understandable[2], and
- which are enough to distinguish the theories we care about.

Warnings:

- The $\leq_B$-structure of the class of all theories is impossibly complex, and
- Proving $\Phi \not\leq_B \Psi$ is extremely difficult in general.

[2] Except in one very important case.

Some Low Complexity Benchmarks

Some "low" isomorphism relations that come up a lot for us:

- 1: There is only one relation with a single class.
- n: For any $n \in \mathbb{N}$, there is only one relation with exactly n classes.
- $\cong_0$: Roughly, a "single natural number" captures each model.
- $\cong_1$: Roughly, a "single real number" captures each model.
- $\cong_2$: Roughly, a "countable set of reals" captures each model.

Not surprisingly:

$$1 <_B 2 <_B 3 <_B \cdots <_B \cong_0 <_B \cong_1 <_B \cong_2 \ \cdots$$

The High Complexity Benchmark

A theory Φ is Borel complete if it is $\leq_B$-maximal among all theories.

That is: for all theories Ψ, $\Psi \leq_B \Phi$.

Theorem (Friedman, Stanley)

Lots of classes are Borel complete:

- *Graphs*
- *Trees*
- *Linear orders*
- *Groups*
- *Fields*
- ...

That's Enough

Surprise: All the theories we investigate today will be exactly equivalent to one of the following:

- $(1, =)$
- $(n, =)$ for some $3 \leq n < \omega$
- $\cong_1$ – real-valued invariants
- $\cong_2$ – set of real invariants

- Borel complete – maximal complexity

Notably:

- No $\cong_0$.
- No need to perform delicate non-embeddability proofs.

Roadmap

O-Minimal Theories

All theories will be first-order, complete, and have an infinite model.

A theory T is o-minimal if $<$ orders the universe and every definable (with parameters) set of elements is a finite union of points and open intervals.

Some examples:

- $(\mathbb{R}, +, \cdot, 0, 1, <)$ is o-minimal (Tarski)
- $(\mathbb{R}, +, \cdot, 0, 1, \exp, <)$ is o-minimal (Wilkie)
- $(\mathbb{R}, +, \cdot, \sin, <)$ is not o-minimal:

  Consider "$\mathbb{Z} = \{x \in \mathbb{R} : \sin(\pi x) = 0\}$"

Why O-Minimal Theories?

The definable subsets (even n-dimensional) of models of o-minimal theories are nice:

- Definable functions are piecewise continuous.
- Definable sets admit cell decompositions.
- Definable sets have Euler characteristics ...
- ... which are preserved under definable injections.
- (and lots more)

Some easy definable sets in $(\mathbb{R}, +, \cdot, 0, 1, <)$:

- $\mathrm{GL}_n(\mathbb{R}) = \{\overline{x} \in \mathbb{R}^{n \times n} : \det(\overline{x}) \neq 0\}$
- The complex field and conjugation function
- S^n
- Projective planes, lens spaces, etc. are *interpretable*

The Divide

The fundamental notion for an o-minimal theory T is whether or not it is locally simple.

Locally here means infinitesimally locally; within a 1-type:

A 1-type is a "complete" consistent intersection of convex definable sets.

Examples of 1-types in RCF:

- The set of "positive infinitesimal" elements (a non-cut)
- The set of "positive infinite" elements (a non-cut)
- The set of "π-like" elements (a cut)

T is locally nonsimple if *at least one* of its types is nonsimple.

Nonsimple Types

A 1-type is nonsimple if there is a non-degenerate definable function from that type to itself.

Examples:

- The set of "positive infinite" elements in RCF is nonsimple under $x \mapsto x + 1$.
- The set of "positive infinitesimal" elements in RCF is nonsimple under $x \mapsto \frac{1}{2}x$.
- The set of "π-like" elements in RCF are nonsimple under $(x, y) \mapsto \frac{1}{2}(x + y)$...

  ...but there is no unary function taking this type to itself.

No Nonsimple Types, I

Theorem

If T is o-minimal and has no nonsimple types, then T is $3^a 6^b$, $\cong_1$, or $\cong_2$, where a is the number of independent non-cuts, and b is the number of independent cuts.

Proof outline, continued:

- If T has no nonsimple types, then countable models $M \models T$ are determined by local behavior: the order types of each 1-type.
- When p is simple:
  - 1 choice of order type for an atomic interval
  - 3 choices of order type for a non-cut
  - 6 choices of order type for a cut

No Nonsimple Types, II

Theorem

If T is o-minimal and has no nonsimple types, then T is $3^a 6^b$, $\cong_1$, or $\cong_2$, where a is the number of independent non-cuts, and b is the number of independent cuts.

Proof outline:

- If a and b are finite, T is $3^a 6^b$
- If a or b is infinite but both are countable, T is $\cong_1$ (real invariants)
- If a or b is uncountable, T is $\cong_2$ (countable sets of real invariants)

The Divide, II

If T is o-minimal and locally simple, there are several values $\cong_T$ can take, but it's essentially a type-counting argument.

If T is o-minimal and locally nonsimple, T turns out to be maximally complicated (Borel complete).

To show this:

1. Find interesting linear orders in models of T, then
2. Use those to show $\mathtt{LO} \leq_B T$

Archimedean Equivalence

Suppose p is a nonsimple type, and a and b realize p.

Say $a \sim b$ if there is some c in $p(M)$, definable over a, where $a \leq b \leq c$ (or reversed if $b \leq a$).

Examples:

- In a real-closed field, two infinite elements a, b have $a \sim b$ if and only if they polynomially bound each other
- In a real additive group, two infinite elements a, b have $a \sim b$ if and only if they linearly bound each other

Fact: $\sim$ is an equivalence relation with convex classes

If $M \models T$, call $p(M)/\sim$ (with its order) the Archimedean ladder of p in M.

Borel Completeness

Theorem

If T is o-minimal and admits a nonsimple type, then T is Borel complete.

Proof outline

- Fix a 1-type p which is nonsimple.
- Linear orders are Borel complete: show $\mathrm{LO} \leq_B T$.
- For any countable $(I, <)$...
- ...let M_I be such that $(p(M_I)/\sim, <)$ is isomorphic to $(I, <)$.
- This is a Borel reduction.

Warning: some details have been skipped for time

Roadmap

Colored Linear Orders

A colored linear order (CLO) is a theory in a language $L = \{<\} \cup \{P_i : i \in I\}$ where

- I is a countable (possibly finite) set,
- Each P_i (a color) is unary, and
- $<$ is a linear order: irreflexive, antisymmetric, transitive, and total

Terminology warning: we do not insist the P_i are disjoint or exhaustive

If T is a CLO and $A \models T$, sometimes refer to A as a CLO as well.

The Theorem

Theorem

If T is a self-additive CLO, T is $\aleph_0$-categorical or Borel complete.

Theorem

For any CLO T:

- *If T is locally simple, T is $(n, =)$, $\cong_1$, or $\cong_2$.*
- *If T is locally nonsimple, T is Borel complete.*

Proof outline:

- Divide T into convex self-additive pieces.
- If one piece is nonsimple, T is Borel complete.
- Each simple piece has a finite number of associated choices.
- If all pieces are simple, the complexity of T is determined by the number of choices.

Self-Additive CLOs

A CLO T is self-additive if it has no nontrivial, convex, definable subsets.

Examples:

- $(\mathbb{Z}, <)$, $(\mathbb{Q}, <)$ and $(\mathbb{R}, <)$ are self-additive:
  They have no proper definable subsets.
- $(\mathbb{R}, \mathbb{Q}, <)$ – the reals with a color for "is rational" – is self-additive:
  The only proper definable sets are $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$.
- $(\mathbb{N}, <)$ is not self-additive:
  $[2, 7]$ is definable (actually every $[m, n]$ is definable).

Fact: if T is self additive, $(I, <)$ is an order, and $\{A_i : i \in I\}$ all model T, then $A_I = \sum_i A_i$ is a model of T and $A_i \prec A_I$ for all i.

Archimedean Equivalence

Let T be self-additive.

If a and b are elements of $A \models T$,say $a \sim b$ if for some formula $\phi(x, y)$:

- $\phi(A, a) = \{x \in A : A \models \phi(x, a)\}$ is convex and bounded
- $\phi(A, a)$ contains both a and b

Theorem (Rubin)

If T is self-additive, then $\sim$ is an equivalence relation with convex classes.

Observation: $\sim$ is preserved under isomorphism, so the quotient order $A/{\sim}$ is an invariant of the model.

Self-Additive CLOs, Complexity I

Lemma

If T is self-additive and $S_1(T)$ is infinite, T is Borel complete.

Proof outline:

- Let $p \in S_1(T)$ be nonisolated.
- Find $M_p \models T$ with one $\sim$-class realizing p.
- For any $(I, <)$, let $M_I = \sum_{i \in I} M_p$.
- The set $M_I^p = \{a \in M_I : \exists b (b \models p \text{ and } a \sim b)\}$ is invariant, and
- $M_I^p / \sim$ is order-isomorphic to I, so
- $I \mapsto M_I$ is a Borel reduction

Self-Additive CLOs, Complexity II

Lemma

If T is a CLO with $S_1(T)$ finite, T is $\aleph_0$-categorical or Borel complete.

Proof by induction on complexity of T – roughly $t = |S_1(T)|$

- If $t = 1$, only $(1, <)$, $(\mathbb{Q}, <)$ or $(\mathbb{Z}, <)$ are possible.
- For $t + 1$, if T is not self-additive, T is a sum of simpler CLOs.
- For $t + 1$, if T is self-additive, T is a shuffle of simpler CLOs.
- If all components are $\aleph_0$-categorical, so is T
- If one component is Borel complete, so is T.

Corollary

All self-additive CLOs are $\aleph_0$-categorical or Borel complete.

Local Behavior

If T is a CLO, there is a space $IT(T)$ of convex types – complete, consistent intersections of convex definable sets.

Think of $IT(T)$ as the infinitesimal decomposition of T.

Example: If T is self-additive, $IT(T)$ is a singleton.

Example: Let $T = \mathrm{Th}(\omega, <) = \{0, 1, 2, 3, 4, \ldots.\}$.

- $IT(T)$ has order type $\omega + 1$.
- The finite pieces n are singletons.
- The final piece is the set of "infinite elements."
  This set is sometimes empty; it depends on the model.

The Divide for CLOs

Let T be some CLO.

Important Facts:

- Every sufficiently saturated model $\mathcal{S}$ has the same $\mathrm{Th}(\Phi(\mathcal{S}))$...
- ...and this theory is self-additive ...
- ...and hence either $\aleph_0$-categorical or Borel complete.

Say T is locally nonsimple if some $\mathrm{Th}(\Phi(\mathcal{S}))$ is Borel complete.
Say T is locally simple if every $\mathrm{Th}(\Phi(\mathcal{S}))$ is $\aleph_0$-categorical.

Easy: if T is locally nonsimple.

General CLOs

Say T is locally simple. Then $\cong_T$ can be characterized:

- [Rosenstein]: $\Phi(\mathcal{S})$ has only finitely many convex subsets up to $\equiv$.
- For any $A \models T$, $\Phi(A)$ is equivalent to a convex subset of $\Phi(\mathcal{S})$.
- $A \models T$ is determined by $\Phi(A)$ for $\Phi \in IT(T)$.

Let n_Φ be the number of forms $\Phi(A)$ can take.
Fact: $n_\Phi > 1$ if and only if Φ is nonisolated.

- If $IT(T)$ is all isolated, T has one countable model
- If $IT(T)$ has finitely many nonisolated points, T has $n > 1$ models.
- If $IT(T)$ has $\aleph_0$ nonisolated points, T is $\cong_1$.
- If $IT(T)$ has $2^{\aleph_0}$ nonisolated points, T is $\cong_2$.

Observe: this is identical in spirit to the o-minimal case.

Wrapup

The general idea is this (for T o-minimal or a CLO):

- Divide T into convex, indivisible pieces
- If T is locally nonsimple then T is Borel complete
- If T is locally simple then the complexity of T is determined essentially on the topology of the type space.

Questions:

- Can the locally complicated / locally simple divide be defined for all ordered theories?
- Does "T is Borel complete or among 1, n, $\cong_1$, $\cong_2$" hold for all ordered theories?