Potential Cardinality

for Countable First-Order Theories

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The Main Idea

The Goal: Understand the countable models of a theory Φ

Chosen framework: if $\Phi \leq_B \Psi$ then the countable models of Φ are "more tame" than the countable models of Ψ.

Relatively easy: show $\Phi \leq_B \Psi$;
Relatively hard: show $\Phi \not\leq_B \Psi$

Theorem (Ulrich, R., Laskowski)

If $\Phi \leq_B \Psi$ then $\|\Phi\| \leq \|\Psi\|$.

Roadmap

Motivation?

Why study Borel reductions?

Comparing the number of models is pretty coarse. Consider:

1. Countable sets of $\mathbb{Q}$-vector spaces
2. Graphs

These both have $\beth_1$ countable models, but
Borel reductions can easily show the former is much smaller than the latter.

Counterexamples to Vaught's conjecture are pretty weird;
Borel reductions give a nice way to make this formal (even given CH).

Borel Reductions

Fix $\Phi, \Psi \in L_{\omega_1\omega}$.

$\mathrm{Mod}_\omega(\Phi)$ and $\mathrm{Mod}_\omega(\Psi)$ are Polish spaces under the formula topology.

$f : \mathrm{Mod}_\omega(\Phi) \to \mathrm{Mod}_\omega(\Psi)$ is a Borel reduction if:

1. For all $M, N \models \Phi$, $M \cong N$ iff $f(M) \cong f(N)$
2. For any $\psi \in L_{\omega_1\omega}$ (with parameters from ω)
   there is a $\phi \in L_{\omega_1\omega}$ (with parameters from ω)
   where $f^{-1}(\mathrm{Mod}_\omega(\Psi \wedge \psi)) = \mathrm{Mod}_\omega(\Phi \wedge \phi)$

   (preimages of Borel sets are Borel)

Say $\Phi \leq_B \Psi$.

A Real Example

Let Φ be "linear orders" and Ψ be "real closed fields." Then $\Phi \leq_B \Psi$.

Proof outline:

- Fix a linear order $(I, <)$
- Pick a sequence $(a_i : i \in I)$ from the monster RCF where $1 \ll a_i$ for all i, and if $i < j$, then $a_i \ll a_j$.
- Let M_I be prime over $\{a_i : i \in I\}$.
- f is "obviously Borel"
- $(I, <) \cong (J, <)$ iff $M_I \cong M_J$.

Establishing Some Benchmarks

Borel reducibility is inherently relative; it's hard to gauge complexity of (the countable models of) a sentence on its own.

One fix is to establish some benchmarks.
The two most important (for us) are:

- Being Borel – a tameness condition which isn't too degenerate
- Being Borel complete – being maximally complicated

Borel Isomorphism Relations

Fix $\Phi \in L_{\omega_1\omega}$. The following are equivalent:

1. Isomorphism for Φ is Borel (as a subset of $\mathrm{Mod}_\omega(\Phi)^2$)
2. There is a countable bound on the Scott ranks of all countable models
3. There is an $\alpha < \omega_1$ where $\equiv_\alpha$ implies $\cong$ for countable models of Φ
4. There is a countable bound on the Scott ranks of all models of Φ
5. There is an $\alpha < \omega_1$ where $\equiv_\alpha$ implies $\equiv_{\infty\omega}$ for all models of Φ.

Fact: if Φ is Borel and $\Psi \leq_B \Phi$, then Ψ is Borel.

Borel Complete Isomorphism Relations

Fix $\Phi \in L_{\omega_1\omega}$. Φ is Borel complete if, for all Ψ, $\Psi \leq_B \Phi$.

Theorem (Friedman, Stanley)

Lots of classes are Borel complete:

- Graphs
- Trees
- Linear orders
- Groups
- Fields
- ...

Fact: If Φ is Borel complete, then Φ is not Borel.

A Serious Question

It's somewhat clear how to show that $\Phi \leq_B \Psi$.

How is it possible to show that $\Phi \not\leq_B \Psi$?

Partial answer: there are some techniques, but they only apply when Φ or Ψ is Borel (and low in the hierarchy).

Very little is known when you can't assume Borel.

Roadmap, II

Back-and-Forth Equivalence

Let M and N be L-structures. $\mathcal{F} : M \to N$ is a back-and-forth system if:

1. $\mathcal{F}$ is a nonempty set of partial functions $M \to N$
2. All $f \in \mathcal{F}$ preserve L-atoms and their negations
3. For all $f \in \mathcal{F}$, all $m \in M$, and all $n \in N$,
   there is a $g \in \mathcal{F}$ where $m \in \mathrm{dom}(g)$, $n \in \mathrm{im}(g)$, and $f \subset g$

Say $M \equiv_{\infty\omega} N$ if there is such an $\mathcal{F}$.

If $M \cong N$ then $M \equiv_{\infty\omega} N$.
If M and N are countable and $M \equiv_{\infty\omega} N$, then $M \cong N$.

Back-and-Forth Equivalence, II

$M \equiv_{\infty\omega} N$ means they are the same from an "intrinsic perspective."

More precisely, the following are equivalent:

- $M \equiv_{\infty\omega} N$
- For every $\phi \in L_{\infty\omega}$, $M \models \phi$ iff $N \models \phi$
- In some $\mathbb{V}[G]$, $M \cong N$
- In every $\mathbb{V}[G]$ making M and N countable, $M \cong N$

The relation "$M \equiv_{\infty\omega} N$" is absolute.

Canonical Scott Sentences

Canonical Scott sentences form a canonical invariant of each $\equiv_{\infty\omega}$-class.
Given an L-structure M, a tuple $\overline{a}$, and an ordinal α, define $\phi_\alpha^{\overline{a}}(\overline{x})$ as follows:

$$\phi_0^{\overline{a}}(\overline{x}) \text{ is } \mathrm{qftp}(\overline{a})$$

$$\phi_\lambda^{\overline{a}}(\overline{x}) \text{ is } \bigwedge_{\beta<\lambda} \phi_\beta^{\overline{a}}(\overline{x}) \text{ for limit } \lambda$$

$$\phi_{\beta+1}^{\overline{a}}(\overline{x}) \text{ is } \phi_\beta^{\overline{a}}(\overline{x}) \wedge \left(\forall y \bigvee_{b\in M} \phi_\beta^{\overline{a}b}(\overline{x}y)\right) \wedge \bigwedge_{b\in M} \exists y \phi_\beta^{\overline{a}b}(\overline{x}y)$$

For some minimal α^*, for all $\overline{a} \in M$, $\phi_{\alpha^*}^{\overline{a}}(\overline{x})$ implies $\phi_{\alpha^*+1}^{\overline{a}}(\overline{x})$.

Define $\mathrm{css}(M)$ as $\phi_{\alpha^*}^{\emptyset} \wedge \bigwedge_{\overline{a}\in M} \forall \overline{x} \phi_{\alpha^*}^{\overline{a}}(\overline{x}) \rightarrow \phi_{\alpha^*+1}^{\overline{a}}(\overline{x})$

Canonical Scott Sentences, II

For all M, N, the following are equivalent:

1. $M \equiv_{\infty\omega} N$
2. $\mathrm{css}(M) = \mathrm{css}(N)$
3. $N \models \mathrm{css}(M)$ (and/or $M \models \mathrm{css}(N)$)

Also, if $|M| = \lambda$, then $\mathrm{css}(M) \in L_{\lambda^+\omega}$.

Also, the relation "$\phi = \mathrm{css}(M)$" is absolute.

Also also, the property "ϕ is in the form of a canonical Scott sentence" is definable and absolute.

Consistency

Proofs in $L_{\infty\omega}$:

- Predictable axiom set
- $\phi, \phi \to \psi \vdash \psi$
- $\{\phi_i : i \in I\} \vdash \bigwedge_{i \in I} \phi_i$

Proofs are now trees which are well-founded but possibly infinite.

$\phi \in L_{\infty\omega}$ is consistent if it does not prove $\neg\phi$.

Warning: folklore

Consistency, II

If $\phi \in L_{\omega_1\omega}$ is formally consistent, then it has a model.

This is not true for larger sentences:

- Let $\psi = \mathrm{css}(\omega_1, <)$, so ψ has no countable models.
- Let $L = \{<\} \cup \{c_n : n \in \omega\}$.
- Let $\phi = \psi \wedge (\forall x \bigvee_n x = c_n)$

Then ϕ is formally consistent, but ϕ has no models.

Fact: the property "ϕ is consistent" is absolute.

Potential Cardinality

Let $\Phi \in L_{\omega_1\omega}$. $\sigma \in L_{\infty\omega}$ is a potential canonical Scott sentence of Φ if:

1. σ has the syntactic form of a CSS
2. σ is formally consistent
3. σ proves Φ

Let $\mathrm{CSS}(\Phi)$ be the set of all these sentences. Let $\|\Phi\| = |\mathrm{CSS}(\Phi)|$.

Easy fact: $I(\Phi, \aleph_0) \leq I_{\infty\omega}(\Phi) \leq \|\Phi\|$.

Note: $I_{\infty\omega}(\Phi)$ is the number of models of Φ up to $\equiv_{\infty\omega}$

A Few Examples

- If T is $\aleph_0$-categorical, $\|T\| = 1$.
- If T is the theory of algebraically closed fields, $\|T\| = \aleph_0$:
  Coded by the transcendence degree: 0, 1, 2, ... or "infinite."
- If $T = (\mathbb{Q}, <, c_q)_{q \in \mathbb{Q}}$, then $\|T\| = \beth_2$.
  Models are coded by which 1-types they realize, and how.

All these examples are grounded – every potential Scott sentence has a model. Weirder examples won't have this property.

Roadmap

The Connection

If $f : \Phi \leq_B \Psi$, then f induces an injection from the countable Scott sentences of Φ to the countable Scott sentences of Ψ.

Theorem (Ulrich, R., Laskowski)

If $f : \Phi \leq_B \Psi$, then get an injection $\overline{f} : \mathrm{CSS}(\Phi) \to \mathrm{CSS}(\Psi)$.

Proof Idea:

- Fix $\tau \in \mathrm{CSS}(\Phi)$.
- $\overline{f}(\tau)$ is what f *would* take τ to, in some $\mathbb{V}[G]$ making τ countable.
- Schoenfield: "$\exists M \in \mathrm{Mod}_\omega(\Phi)\ (M \models \tau \wedge f(M) \models \sigma)$" is absolute
- If G_1 and G_2 are independent, then $\mathbb{V}[G_1] \cap \mathbb{V}[G_2] = \mathbb{V} \ldots$
- $\ldots$ so $\overline{f}(\tau) \in \mathbb{V}$ and $\overline{f}(\tau) \in \mathrm{CSS}(\Psi)$.

Some Easy Facts

Fact: If Φ is Borel, then $\|\Phi\| < \beth_{\omega_1}$
Proof Idea:

- (Hjorth): If Φ is Π^0_α, then Φ is reducible to $\cong_\alpha$.
- $\| \cong_\alpha \| = \beth_{-1+\alpha+1}$, so $\|\Phi\| \leq \beth_{-1+\alpha+1}$.

Fact: If Φ is Borel complete, then $\|\Phi\| = \infty$
Proof Idea:

- (Folklore): all ordinals are back-and-forth inequivalent, so $\|\mathrm{LO}\| = \infty$.
- $\mathrm{LO} \leq_B \Phi$, so $\|\Phi\| = \infty$.

Some Excellent Questions

Hanf Number: Is it possible to get $\beth_{\omega_1} \leq \|\Phi\| < \infty$?
Unknown!

Is it possible for $\|\Phi\| = \infty$ when Φ is not Borel complete?
Yes!
Unknown if there are first-order examples

Is it possible for $\|\Phi\| < \beth_{\omega_1}$ when Φ is not Borel?
Yes! And there are first-order examples!

The last "yes!" answers a stubborn conjecture:
Can a first-order theory be neither Borel nor Borel complete?

One Answer

Theorem (Friedman, Stanley)

Let Φ be the sentence describing abelian p-groups, for some prime p. Then Φ is not Borel and not Borel complete. Also, $\|\Phi\| = \infty$.

Proof Sketch

- Can construct p-groups of arbitrary (ordinal) Ulm height, so $\|\Phi\| = \infty$, so Φ is not Borel.
- Can't embed countable sets of reals into Φ:
  Suitably generic sets of reals go to the same group, so injectivity fails.

So it is possible for Φ to be neither Borel nor Borel complete.
What about for a first-order theory?

Three First Order Examples

We worked with three complete first-order theories: REF, K, and TK.

REF is superstable, classifiable (depth 1), and not $\aleph_0$-stable.
$\|\mathrm{REF}\| = \beth_2$, so REF is not Borel complete, but REF is not Borel.

K is $\aleph_0$-stable and classifiable (depth 2).
$\|\mathrm{K}\| = \beth_2$, so K is not Borel complete, but K is not Borel.

TK is $\aleph_0$-stable and classifiable (depth 2).
TK is Borel complete, so $\|\mathrm{TK}\| = \infty$, but $I_{\infty\omega}(\mathrm{TK}) = \beth_2$.

REF is grounded; TK is not; groundedness of K is open.

Refining Equivalence Relations

REF is in the following language: $L = \{E_n : n \in \omega\}$. REF states:

1. Each E_n is an equivalence relation, all classes infinite
2. E_n has exactly 2^n classes
3. Each E_n class refines into exactly E_{n+1} classes

REF is superstable but not $\aleph_0$-stable (type counting).

In fact REF is super nice from a stability-theory perspective.

REF has Many Countable Models

We can embed "countable sets of reals" into $\mathrm{Mod}_\omega(\mathrm{REF})$.

Proof sketch:

- Pretend we have names from 2^n for each E_n class
- Then we have names from 2^ω for each E_∞ class
- Any *dense* $X \subset 2^\omega$ can be the set of E_∞ class we actually realize (say, realize them infinitely many times)

- Coding trick: we can realize certain E_∞ classes finitely many times, so that we still get this naming

So $\cong_2 \leq_B \mathrm{REF}$ and $I_{\infty\omega}(\mathrm{REF}) \geq \beth_2$

REF is Grounded

Recall: Φ is grounded if everything in $\mathrm{CSS}(\Phi)$ has a model.

Theorem: Let $\phi \in \mathrm{CSS}(\mathrm{REF})$. Then ϕ has a model.
Proof sketch:

- Let $\mathbb{V}[G]$ think ϕ is countable, so it has a model
- The countable model M of ϕ is *unique* up to isomorphism

- Compute a bunch of invariants of M in $\mathbb{V}[G]$
- Even if $M \notin \mathbb{V}$, all the invariants are in $\mathbb{V}$

- In $\mathbb{V}$, build a model $N \models \mathrm{REF}$
- In $\mathbb{V}[G]$, show $M \equiv_{\infty\omega} N$, so that $N \models \phi$ in $\mathbb{V}$

Note: the invariants are essentially a tree of Scott sentences extending ϕ, in a larger language, plus some related trees

REF is not Borel Complete

Theorem: $I_{\infty\omega}(\mathrm{REF}) = \beth_2$
Proof sketch:

- We already know $I_{\infty\omega}(\mathrm{REF}) \geq \beth_2$

- Let $M \models \mathrm{REF}$ be arbitrary.
- Let $N \subset M$ drop all but a countable subset of each E_∞ class
- $|N| \leq \beth_1$ and $M \equiv_{\infty\omega} N$.

- There are at most $\beth_2$ models of size $\beth_1$, up to $\equiv_{\infty\omega}$
- So $I_{\infty\omega}(\mathrm{REF}) \leq \beth_2$

Corollary: $\|\mathrm{REF}\| = \beth_2$
Corollary: REF is not Borel complete

So Far, So Normal

What we know so far:

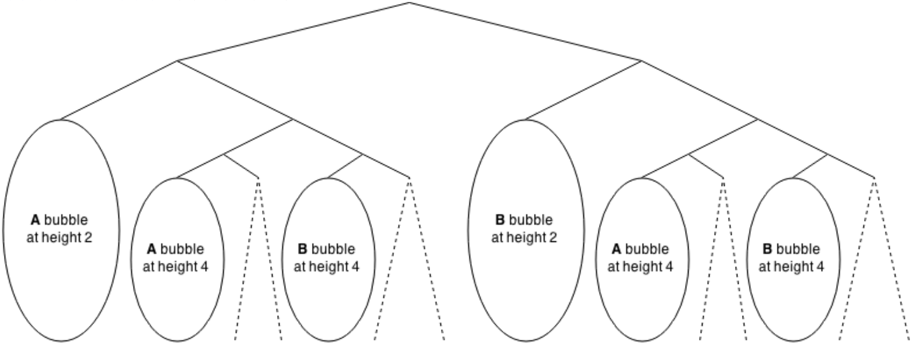
- REF is tame, from a stability-theory perspective
- REF is grounded
- $\|\mathrm{REF}\| = I_{\infty\omega}(\mathrm{REF})$, and both are a reasonable, small number
- REF is not Borel complete

Everything right now makes REF look very well-behaved.

REF is Not Borel

REF has countable models of arbitrarily high Scott ranks.

The Construction: Fix $A, B \models \mathrm{REF}$ countable where $A \equiv_\alpha B$ and $A \not\cong B$. Fix $X \subset 2^\omega$ countable and dense. Construct M_X as a branching balanced bubble model as follows:



Realize the E_∞-class of $\eta \in 2^\omega$ iff $\eta \in X$.

If $X \neq Y$, then $M_X \equiv_{\alpha+1} M_Y$ but $M_X \not\cong M_Y$.

Wrapup on REF

Thus REF is an example of the following:

- A complete first order theory in a countable language, where
- The isomorphism relation is not Borel, and
- The isomorphism relation is not Borel complete

More importantly: potential cardinality gives a way to show the nonexistence of a Borel reduction, even when the underlying isomorphism relation is not Borel.

Side benefit: the proof was model-theoretic, rather than forcing-theoretic.

Note: after naming $\mathrm{acl}(\emptyset)$, the theory is Borel (in fact Π^0_3).

Koerwien's Example

The theory K is in the language $L = \{U, C_n, V_n, S_n, \pi_n : n \in \omega\}$. K states:

- U and each of the V_n are infinite sorts; C_n is a sort of size two
- $\pi_n : V_n : U \times C_0 \times \cdots C_n$ is a surjection
- $S_n : V_n \to V_n$ is a successor function
- $\pi_n \circ S_n = \pi_n$

K is $\aleph_0$-stable, classifiable, and has depth two

K is not Borel, but $\|\mathrm{K}\| = I_{\infty\omega}(\mathrm{K}) = \beth_2$; K may not be grounded;
It is unknown if K and REF are $\leq_B$-comparable

Note: $\mathrm{Aut}(\mathrm{acl}(\emptyset))$ is $(2^\omega, +)$, which is abelian;
after naming $\mathrm{acl}(\emptyset)$, isomorphism is Π^0_3

The Koerwien Tweak

The theory TK is in the language $L = \{U, C_n, V_n, S_n, \pi_n, p_n : n \in \omega\}$. TK states:

- U and each of the V_n are infinite sorts; C_n is a sort of size 2^n
- $\pi_n : V_n \to U \times C_n$ is a surjection
- $p_n : C_{n+1} \to C_n$ is a two-to-one surjection
- $S_n : V_n \to V_n$ is a successor function
- $\pi_n \circ S_n = \pi_n$

TK is $\aleph_0$-stable, classifiable, and has depth two

TK is Borel complete, but $I_{\infty\omega}(\mathrm{TK}) = \beth_2$, so not Borel and not grounded

Note: the only difference between TK and K is $\mathrm{Aut}(\mathrm{acl}(\emptyset))$;
Here $\mathrm{Aut}(\mathrm{acl}(\emptyset))$ is $\mathrm{Aut}(2^{<\omega}, <)$, which is highly nonabelian
After naming $\mathrm{acl}(\emptyset)$, K and TK become equivalent (so Π^0_3)

The End

Thank you!

The paper in question:
arXiv:1510.05679