Borel-Complete O-Minimal Theories

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Isomorphisms

Question

Given two countable models $\mathcal{M}$ and $\mathcal{N}$ of a sentence $\Phi \in L_{\omega_1,\omega}$, how hard is it to tell if $\mathcal{M} \cong \mathcal{N}$?

Sometimes it's easy:

- Φ is the theory of an infinite set
- Φ is $\mathrm{Th}(\mathbb{Q}, <, c_n)_{n\in\omega}$

Sometimes it's hard:

- Φ is the theory of graphs (or groups, or fields, or. . . .)
- Φ is $\mathrm{Th}(\mathbb{Z}, <)$

Isomorphism and Borel Spaces

Give $X = \mathrm{Mod}(\omega, \Phi)$ the formula topology, so $\cong$ is a subset of $X \times X$.

If $\cong$ is a *Borel* set, there is a "computable" invariant that classifies models.

If $\cong$ is *Borel-complete* then there is no such invariant, and this as difficult as any isomorphism problem could possibly be.

It turns out that if Φ is an o-minimal first-order theory, then there is a dichotomy: $\cong$ is either Borel (and very low) or Borel-complete.

Nonsimplicity

From now on, T is a countable o-minimal theory.

Definition (Mayer)

A type $p \in S_1(A)$ is *nonsimple* if there is a non-degenerate A-definable function $f : p^n \to p$, for some n.

Some examples of nonsimple types:

- $(\mathbb{R}^{\mathrm{alg}}, <, +, \cdot)$: the type "at infinity" is nonsimple: $x \mapsto x^2$
- $(\mathbb{Q}, <, +, c_q)_{q \in \mathbb{Q}}$: the type "at π" is nonsimple: $(x, y) \mapsto \frac{1}{2}(x + y)$
- $(\mathbb{Z}, <)$: the type "$x = x$" is nonsimple: $x \mapsto x + 1$

The Main Theorem

Theorem (Sahota, 2013)

- *If, for some finite A, there is a nonsimple type over A, then for some finite $B \supset A$, T_B is Borel-complete.*
- *If there is no such A, then T is not Borel-complete – it's Π^0_3.*

But, to answer the question for T:

Theorem (R. 2014)

If T admits a nonsimple type over some finite A, then T is Borel-complete.

Examples

Most interesting o-minimal theories are Borel-complete:

Corollary

Any nontrivial o-minimal theory is Borel-complete. In particular, any o-minimal theory which defines an infinite group is Borel-complete.

Corollary

Any discretely ordered o-minimal theory is Borel-complete.

The paradigmatic non-Borel-complete o-minimal theory is $(\mathbb{Q}, <)$ with some countable set of constants added.

No Nonsimple Types

Theorem (Mayer)

Suppose T has no nonsimple types over any finite set. Let $\mathcal{M}, \mathcal{N} \models T$ be countable. Then $\mathcal{M} \cong \mathcal{N}$ if and only if, for all $p \in S_1(\emptyset)$, $p(\mathcal{M}) \cong p(\mathcal{N})$.

Corollary (Sahota)

If T has no nonsimple types over any finite set, then $(Mod(T), \cong)$ is Π^0_3.

Archimedean Equivalence

For the case where there *is* a nonsimple type, we want to embed $(\mathrm{LO}, \cong)$ into $(\mathrm{Mod}(T), \cong)$. This is how we do it:

Definition

In a nonsimple type $p \in S_1(A)$, if a and b realize p, say $a \sim b$ if $a < b$ and there is $a' \in \mathrm{cl}_A^p(a)$ where $b \leq a'$.

$\sim$ is an equivalence relation whose classes are convex, so $p(\mathcal{M})/\sim$ is a linear order, called the *ladder*.

Some examples:

- Give $\mathcal{M}_1 = (\mathbb{Q}^n, +, <)$ the lexicographic order. Then $p = \{x > 0\}$ is a complete type and $p(\mathcal{M}_1)/\sim$ is $\{[e_1] > \ldots > [e_n]\}$.
- If L is a linear order, let $\mathcal{M}_2 = (L \times \mathbb{Z}, <)$. So $p(x) = \{x = x\}$ is a complete type and $p(\mathcal{M}_2)/\sim$ is isomorphic to L.

A Nonsimple Type

Fix a nonsimple type $p \in S_1(\emptyset)$. We build a Borel reduction $(\mathrm{LO}, \cong) \to (\mathrm{Mod}(T), \cong)$ where L appears as the ladder $p(\mathcal{M}_L)/\sim$ as follows:

1. For a countable linear order L, fix a set $X_L = \{x_\alpha : \alpha \in L\}$ from p, where if $\alpha < \beta$, then $[x_\alpha] < [x_\beta]$ in the Archimedean sense.
2. Let $\mathcal{M}_L \models T$ be prime over X_L.
3. The map $\alpha \mapsto [x_\alpha]$ *should be* an order-isomorphism $L \to p(\mathcal{M}_L)/\sim$.
4. Then for any orders L and L', we have $L \cong L'$ iff $\mathcal{M}_L \cong \mathcal{M}_{L'}$.

The tricky spot is surjectivity in (3); say that p is *faithful* if this map is surjective (in which case T is Borel-complete).

Unfaithful Types

If p is nonsimple and nonisolated, then there is always a faithful type somewhere. In particular, non-cuts are faithful.

Example

Let $T = \mathrm{Th}(Q, <, f)$, where $f(x, y, z) = x + y - z$. Then $x = x$ is an isolated, nonsimple type which is not faithful.

If we pick two parameters (call them 0 and 1) then we get a nonsimple (faithful) type at “infinity” where we can build a ladder. But it will *not* be preserved under isomorphism.

Tails

Example

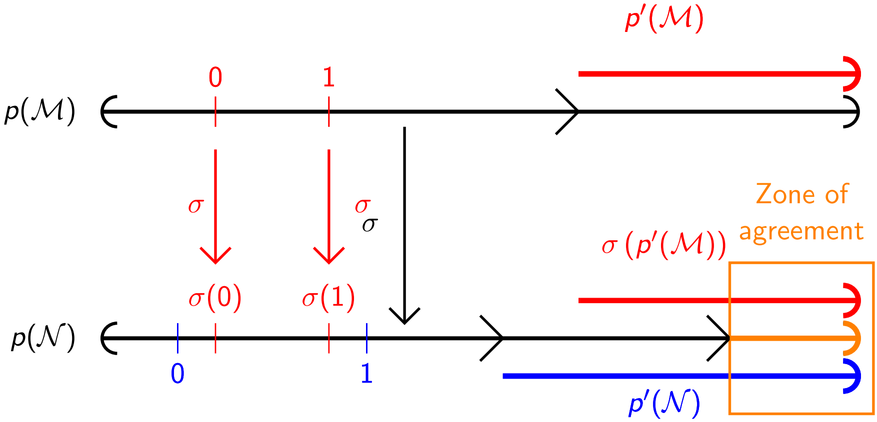
Let $T = \mathrm{Th}(Q, <, f)$, where $f(x, y, z) = x + y - z$. Then $x = x$ is an isolated, nonsimple type which is not faithful.

Given two parameter choices $\{0, 1\}$ and $\{0', 1'\}$, if x and y are "big enough" – infinite with respect to $\{0, 0', 1, 1'\}$ – then x and y are equivalent over $(0, 1)$ if and only if they're equivalent over $(0', 1')$.

So if we fix $\{0, 1\}$, then build a long enough ladder above them, a *tail* of our intended linear order is preserved under isomorphism.

The Tail Picture

So suppose p is our nonsimple type, and $\sigma : \mathcal{M} \to \mathcal{N}$ is an isomorphism. Then the situation looks like this:



So there is a common *tail* of the two linear orders.

The General Proof

Fix a nonsimple type $p \in S_1(\emptyset)$. We would like build a Borel reduction from the LO to $(\mathrm{Mod}(T), \cong)$ as follows:

1. Fix a set $A = \{0, \ldots, n\}$ of parameters to get a nonisolated type.
2. For a countable linear order L, fix a set $X_L = \{x_\alpha : \alpha \in L\}$ from p, where $x_\alpha > \mathrm{cl}^p_A(X_\alpha)$.
3. Let $\mathcal{M}_L \models T$ be prime over $A \cup X_L$.
4. The map $\alpha \mapsto [x_\alpha]$ *will be* an order-isomorphism $L \to p(\mathcal{M}_L)/\sim_A$.
5. Then for any orders L and L', if $\mathcal{M}_L \cong \mathcal{M}_{L'}$, then L and L' are isomorphic *on a tail*.

The Last Piece

Lemma

There is an invariant, Borel-complete subclass $\mathcal{S} \subset LO$ where $\cong$ is equivalent to tail isomorphism.

So if T admits a nonsimple type, then T is Borel-complete.

Thank you!